Self study	**To be handed in**
Read pp. 41-45 Examples 3.2.1, 3.2.2 Exercise 3A: 1, 2	Exercise 3A: 2, 7

3. Centre of mass

- Able to find the centre of mass for objects made up of parts whose centres of mass you already know
- Understand how the procedure is justified by the theory of moments
- Know the weighted mean formula for finding centres of mass
- Know how to use the centre of mass to determine equilibrium positions for objects

3.1 One-dimensional objects

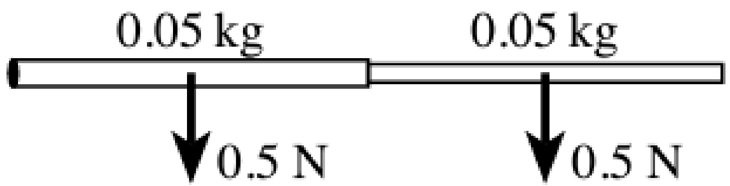
Example 3.1.1

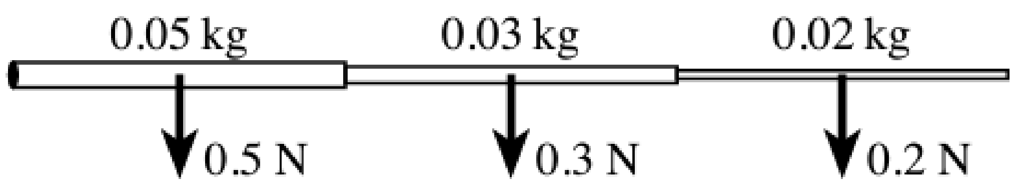
A portable radio has a telescopic aerial, consisting of three parts of masses 0.05 kg, 0.03 kg and 0.02 kg, each of length 20 cm. The first part is hinged to the radio at one end, and the other two parts slide inside it. All three parts are uniform. Find the distance of the centre of mass from the hinge when

(a) the aerial is closed up,

(b) the second and third parts are pulled out together to make an aerial of length 40 cm,

(c) the aerial is fully extended.





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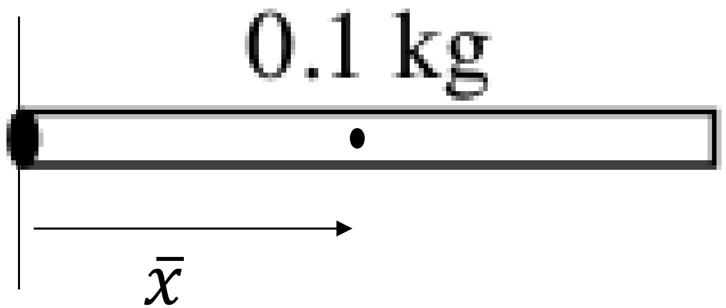
(a) the aerial is closed up,

$$\bar{x} = 0.1\text{ m}$$



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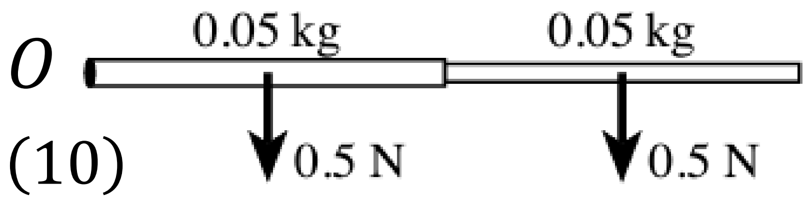
(b) the second and third parts are pulled out together to make an aerial of length 40 cm,



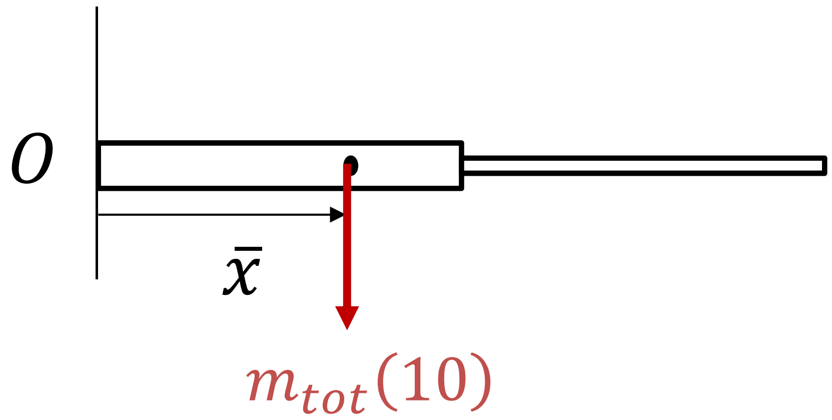
$$M_O = 0.1(0.05)(10) + 0.3(0.05)(10)$$

$$M_O = 0.2 \text{ Nm}$$

$$= \bar{x} m_{tot}(10)$$

$$\therefore \bar{x} m_{tot}(10) = 0.2$$

$$\therefore \bar{x} = \frac{0.2}{m_{tot}(10)} = \frac{0.2}{0.1(10)} = 0.2 \text{ m}$$

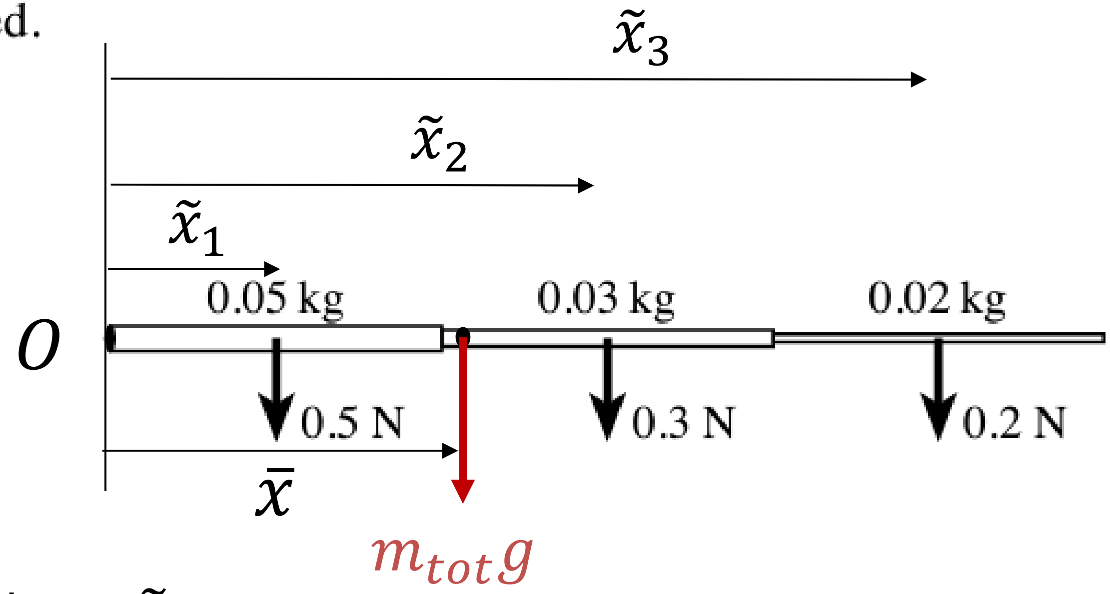


Example 3.1.1

A portable radio has a telescopic aerial, consisting of three parts of masses 0.05 kg, 0.03 kg and 0.02 kg, each of length 20 cm. The first part is hinged to the radio at one end, and the other two parts slide inside it. All three parts are uniform. Find the distance of the centre of mass from the hinge when

(c) the aerial is fully extended.

$$M_O = \bar{x} m_{tot} g$$



$$M_O = m_1\tilde{x}_1 g + m_2\tilde{x}_2 g + m_3\tilde{x}_3 g$$

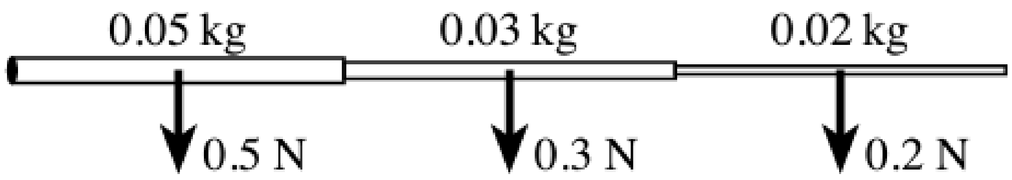
$$\therefore \bar{x} m_{tot} g = m_1\tilde{x}_1 g + m_2\tilde{x}_2 g + m_3\tilde{x}_3 g$$

$$\therefore \bar{x} = \frac{m_1\tilde{x}_1 + m_2\tilde{x}_2 + m_3\tilde{x}_3}{m_{tot}} = \frac{m_1\tilde{x}_1 + m_2\tilde{x}_2 + m_3\tilde{x}_3}{m_1 + m_2 + m_3}$$

If an object is made up of n sections of masses m_1, m_2, … , m_n, each with its centre of mass on a line and having coordinates x_1, x_2, … , x_n, then the centre of mass has coordinate $\bar{x}$ where

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{M} \text{ and } M = m_1 + m_2 + \ldots + m_n .$$

Mass	m_1	m_2	m_3	$\ldots$	m_n	M
Distance	x_1	x_2	x_3	$\ldots$	x_n	$\bar{x}$



Mass (kg)	0.05	0.03	0.02	0.1
Distance (cm)	10	30	50	$\bar{x}$

$$0.05 \times 10 + 0.03 \times 30 + 0.02 \times 50 = 0.1\bar{x} \,,$$

so $$\bar{x} = \frac{2.4}{0.1} = 24 \,.$$

3.2 Two-dimensional objects

If an object is made up of n sections of masses m_1, m_2, … , m_n, each with its centre of mass in a plane and having coordinates (x_1, y_1), (x_2, y_2), … , (x_n, y_n), then the centre of mass has coordinates $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{M}, \qquad \bar{y} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{M}$$

and $M = m_1 + m_2 + \ldots + m_n$.

If the centres of mass of the sections have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, … , $\mathbf{r}_n$, then the centre of mass has position vector $\bar{\mathbf{r}}$, where

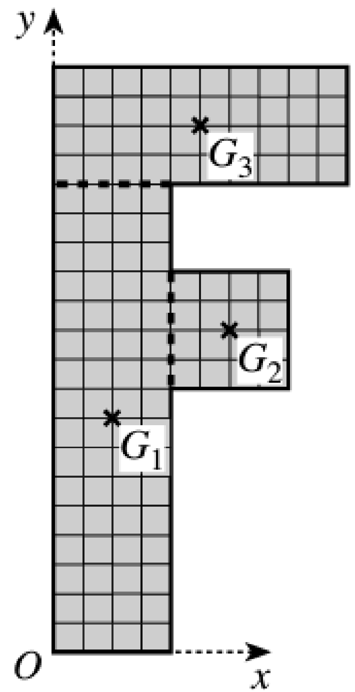
$$\bar{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{M}.$$

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \bar{\mathbf{r}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

Mass				
x-coordinate				
y-coordinate				



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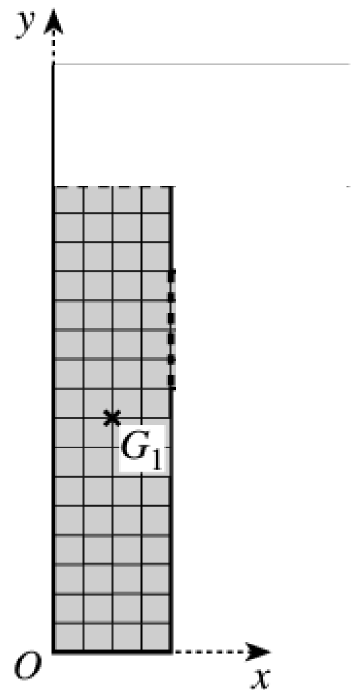
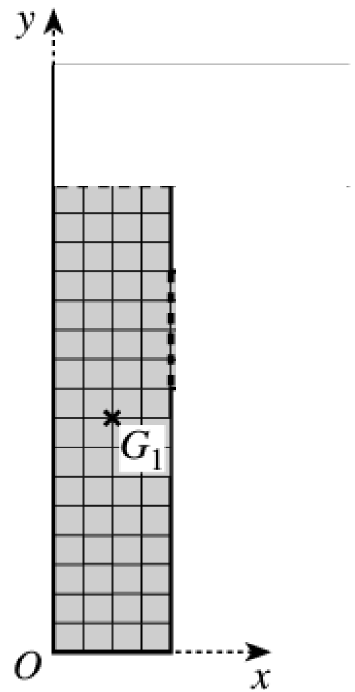
Mass				
x-coordinate				
y-coordinate				



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

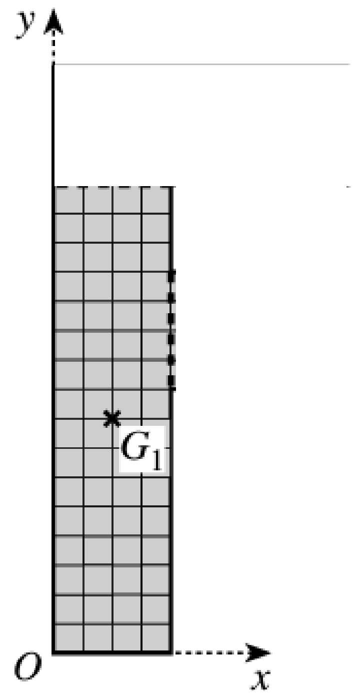
Mass	$64k$			
x-coordinate				
y-coordinate				



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

Mass	$64k$			
x-coordinate	2			
y-coordinate	8			



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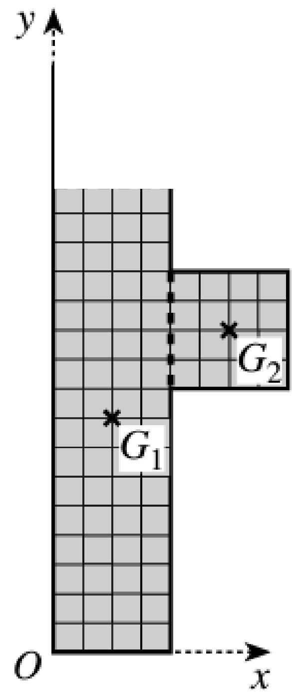
A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

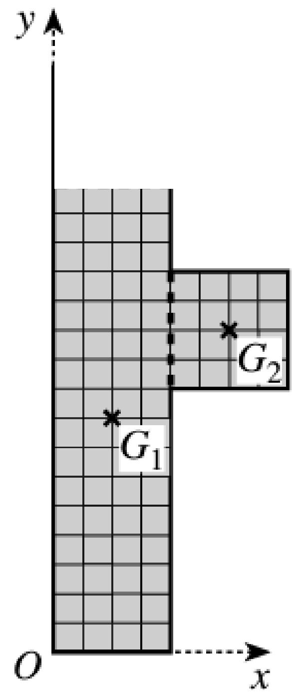
Mass	$64k$			
x-coordinate	2			
y-coordinate	8			



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

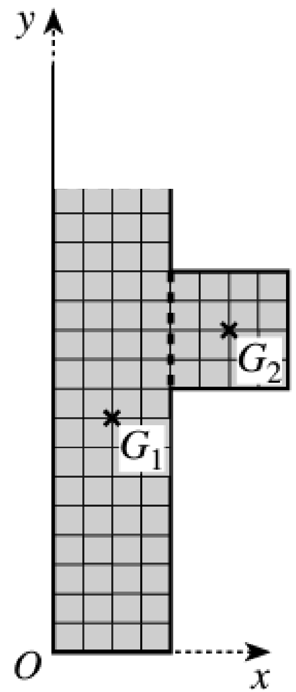
Mass	$64k$	$16k$		
x-coordinate	2			
y-coordinate	8			



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

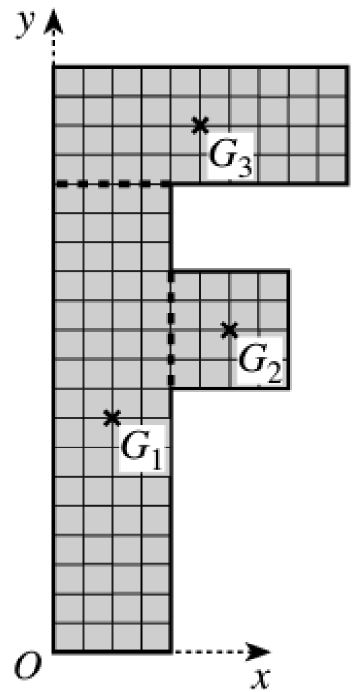
Mass	$64k$	$16k$		
x-coordinate	2	6		
y-coordinate	8	11		



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

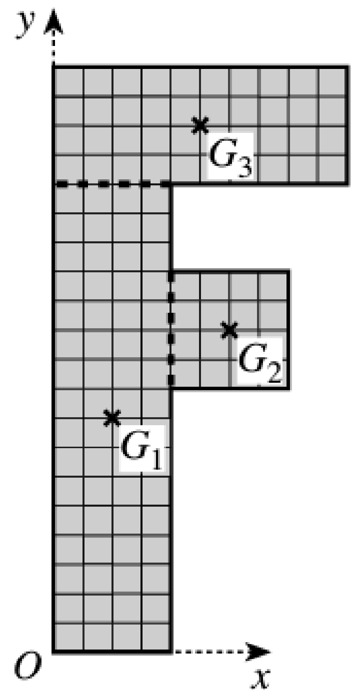
Mass	$64k$	$16k$		
x-coordinate	2	6		
y-coordinate	8	11		



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

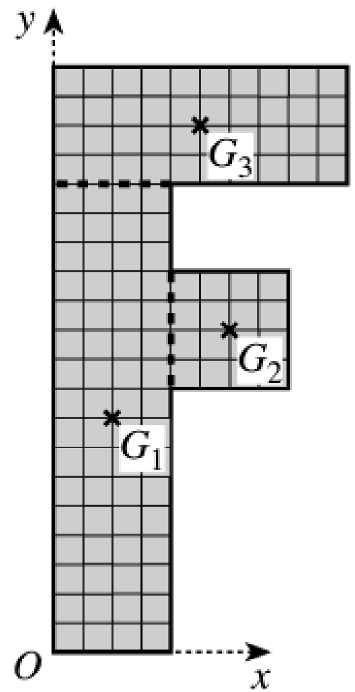
Mass	$64k$	$16k$	$40k$	
x-coordinate	2	6	5	
y-coordinate	8	11	18	



Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

Mass	$64k$	$16k$	$40k$	$120k$
x-coordinate	2	6	5	$\bar{x}$
y-coordinate	8	11	18	$\bar{y}$



$$\bar{x} = \frac{m_1\tilde{x}_1 + m_2\tilde{x}_2 + m_3\tilde{x}_3}{m_1 + m_2 + m_3}$$

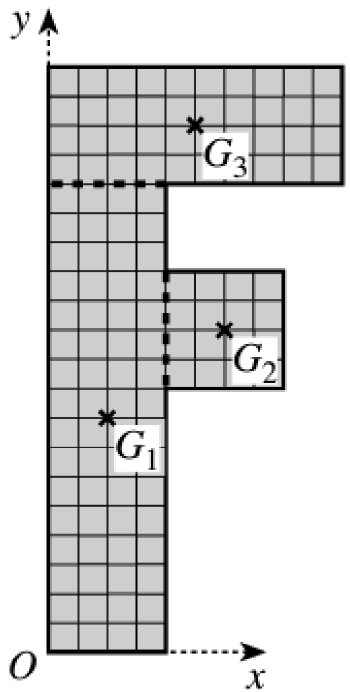
$$\bar{x} = \frac{64k(2) + 16k(6) + 40k(5)}{64k + 16k + 40k}$$

$$\bar{x} = \frac{53}{15}$$

Example 3.2.1

A letter F is drawn on graph paper, as in Fig. 3.6, and a piece of thin card is then cut out to this shape. Find the centre of mass of the card.

Mass	$64k$	$16k$	$40k$	$120k$
x-coordinate	2	6	5	$\bar{x}$
y-coordinate	8	11	18	$\bar{y}$



$$\bar{y} = \frac{m_1\tilde{y}_1 + m_2\tilde{y}_2 + m_3\tilde{y}_3}{m_1 + m_2 + m_3}$$

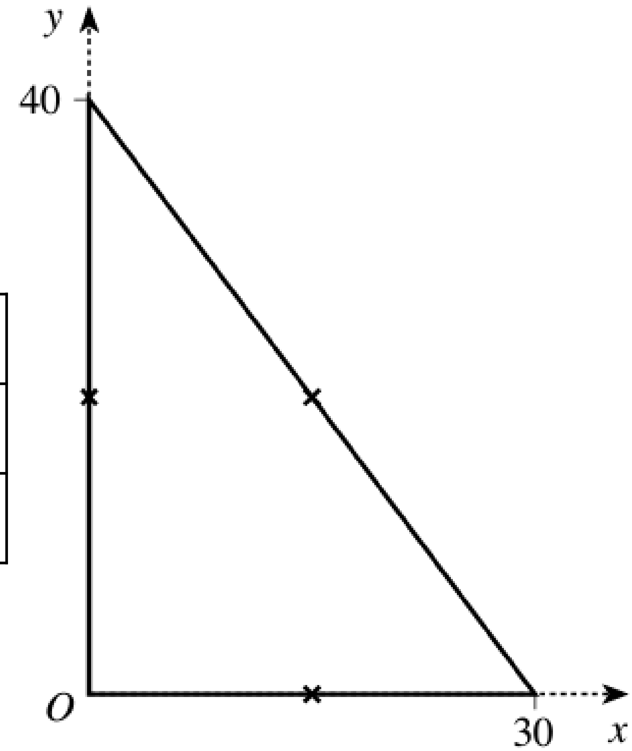
$$\bar{y} = \frac{64k(8) + 16k(11) + 40k(18)}{64k + 16k + 40k}$$

$$\bar{y} = \frac{176}{15} \qquad (\bar{x}, \bar{y}) = \left(\frac{53}{15}, \frac{176}{15}\right)$$

Example 3.2.2

A thin wire of uniform thickness is bent into a triangle with sides of length 30 cm, 40 cm and 50 cm. Find the position of the centre of mass.

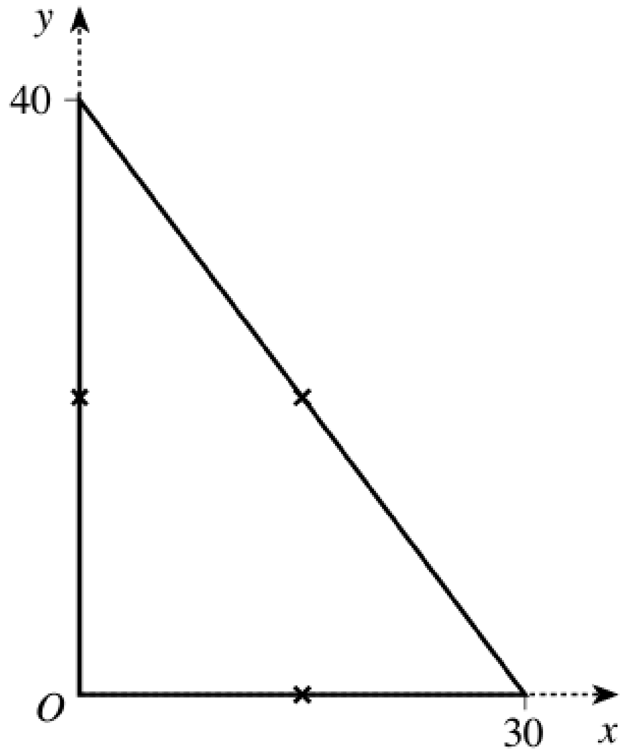
Mass				
x				
y				



Example 3.2.2

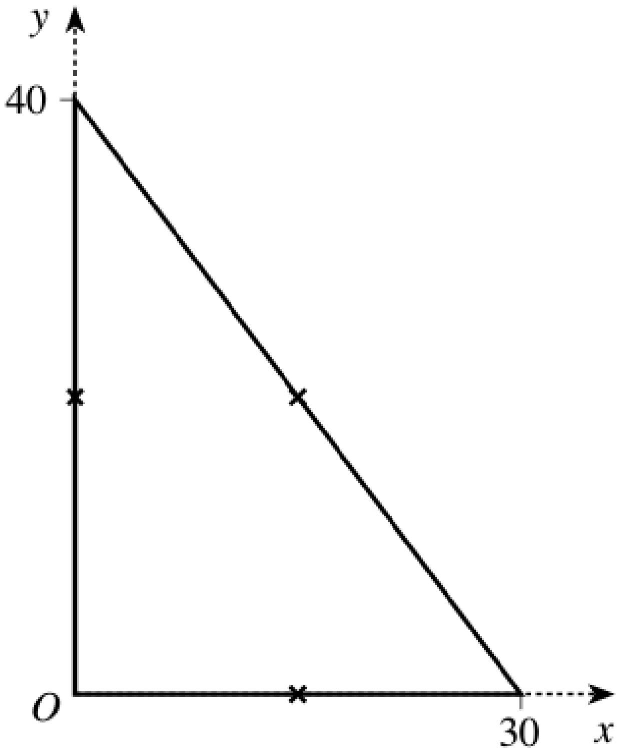
A thin wire of uniform thickness is bent into a triangle with sides of length 30 cm, 40 cm and 50 cm. Find the position of the centre of mass.

Mass	$30k$			
x	15			
y	0			



Example 3.2.2

A thin wire of uniform thickness is bent into a triangle with sides of length 30 cm, 40 cm and 50 cm. Find the position of the centre of mass.

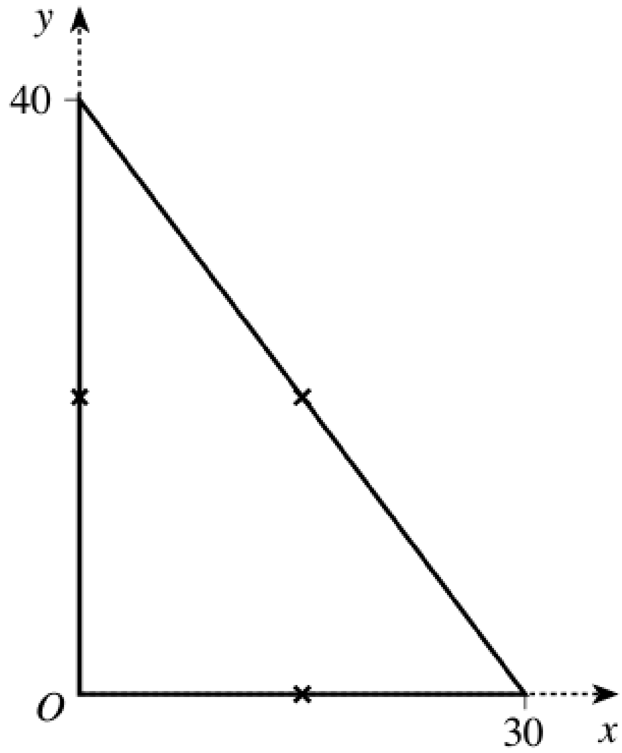


Mass	$30k$	$40k$		
x	15	0		
y	0	20		

Example 3.2.2

A thin wire of uniform thickness is bent into a triangle with sides of length 30 cm, 40 cm and 50 cm. Find the position of the centre of mass.

Mass	$30k$	$40k$	$50k$	
x	15	0	15	
y	0	20	20	



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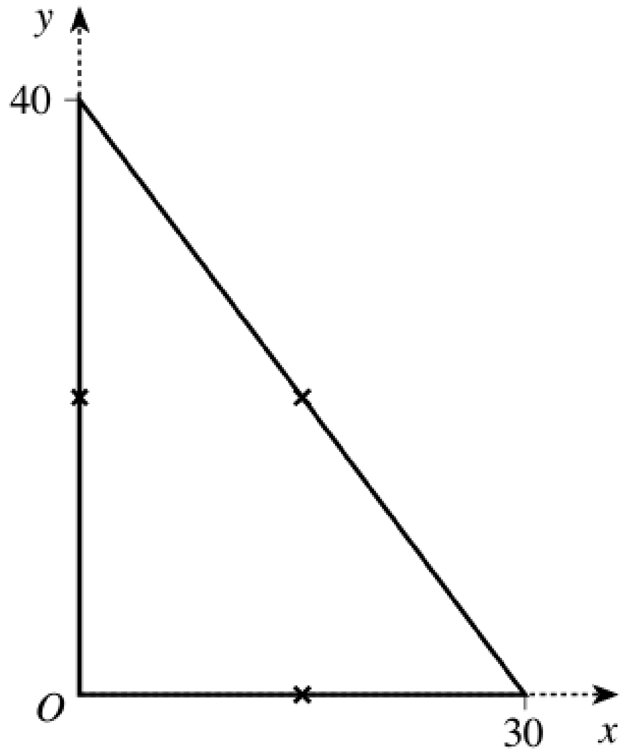
A thin wire of uniform thickness is bent into a triangle with sides of length 30 cm , 40 cm and 50 cm . Find the position of the centre of mass.



Mass	$30k$	$40k$	$50k$	$120k$
x	15	0	15	$\bar{x}$
y	0	20	20	$\bar{y}$

$$\bar{x} = \frac{30k(15) + 40k(0) + 50k(15)}{120k}$$

$$\bar{x} = 10 \text{ cm}$$

$$\bar{y} = \frac{30k(0) + 40k(20) + 50k(20)}{120k} = 15 \text{ cm}$$

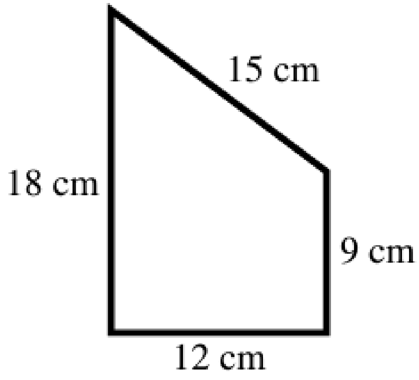
$$(\bar{x}, \bar{y}) = (10{,}15) \text{ cm}$$

Class exercises: Exercise 3A pg. 48

10 A frame, in the form of the quadrilateral shown in the diagram, is made from a piece of uniform wire of length 54 cm . Find the distance of the centre of mass of the frame from

(a) the side of length 18 cm ,

(b) the side of length 12 cm .



Mass					
x					
y					

Class exercises: Exercise 3A pg. 48

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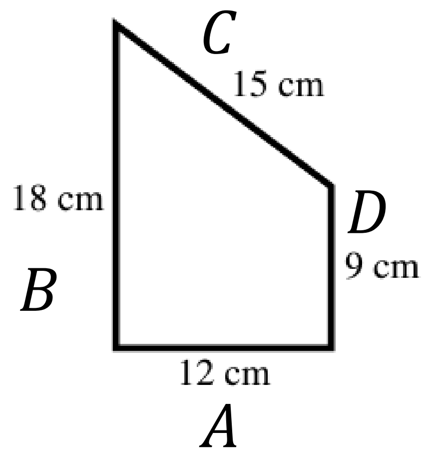

	A	B	C	D	
Mass					
x					
y					

Class exercises: Exercise 3A pg. 48

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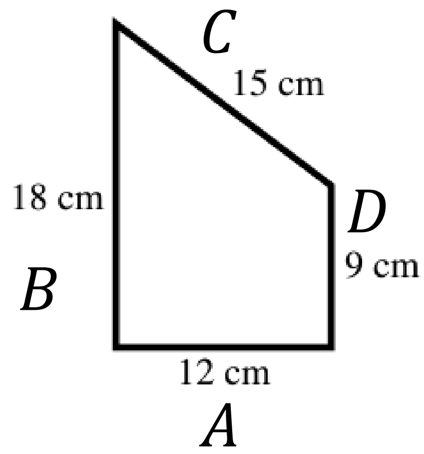

	A	B	C	D	
Mass	$12k$				
x	6				
y	0				

Class exercises: Exercise 3A pg. 48

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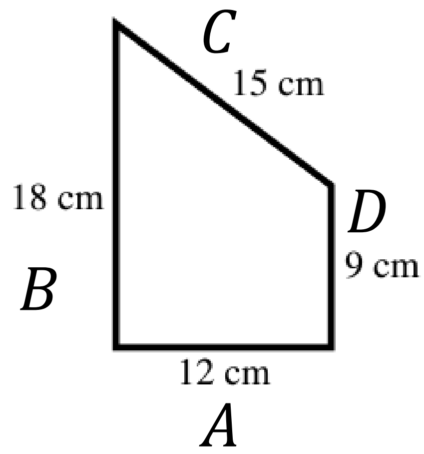

	A	B	C	D	
Mass	$12k$	$18k$	$15k$	$9k$	$54k$
x	6	0	6	12	$\bar{x}$
y	0	9	13.5	4.5	$\bar{y}$

$$\bar{x} = \frac{12k(6) + 18k(0) + 15k(6) + 9k(12)}{54k} = 5 \text{ cm}$$

Class exercises: Exercise 3A pg. 48

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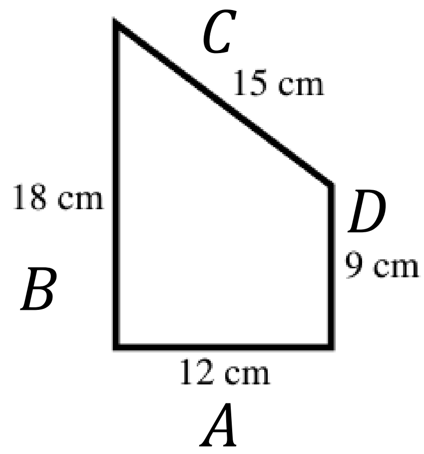
(a) the side of length 18 cm,

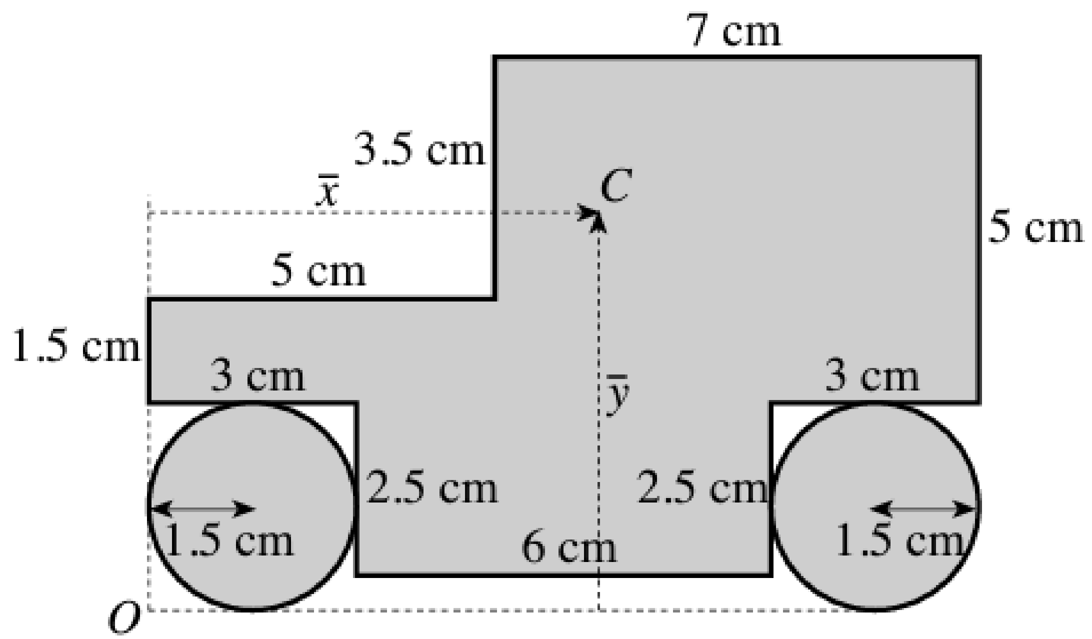
(b) the side of length 12 cm.



	A	B	C	D	
Mass	$12k$	$18k$	$15k$	$9k$	$54k$
x	6	0	6	12	$\bar{x}$
y	0	9	13.5	4.5	$\bar{y}$

$$\bar{y} = \frac{12k(0) + 18k(9) + 15k(13.5) + 9k(4.5)}{54k} = 7.5 \text{ cm}$$

16 A sheet of metal has uniform thickness and uniform density. The shape shown in the diagram is cut from the sheet of metal. Find the coordinates $(\bar{x}, \bar{y})$ of the centre of mass C of the shape.



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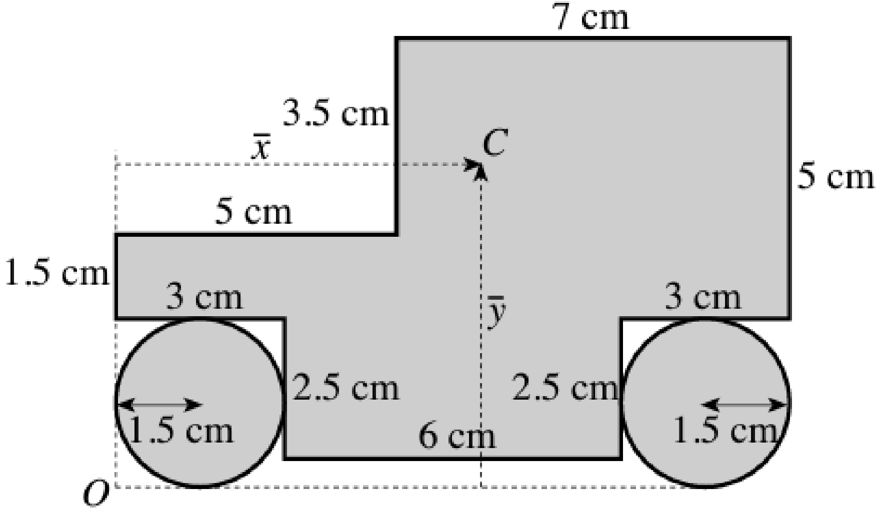

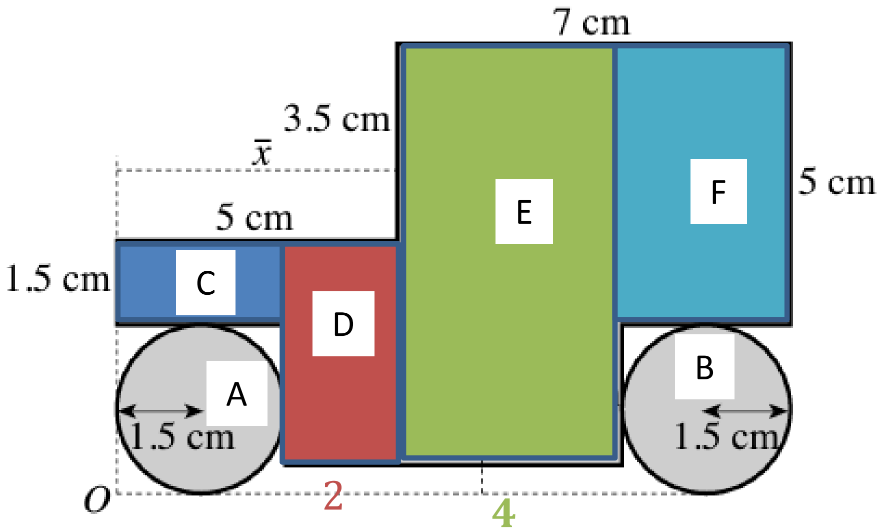

	A	B	C	D	E	F	
m							
x							
y							

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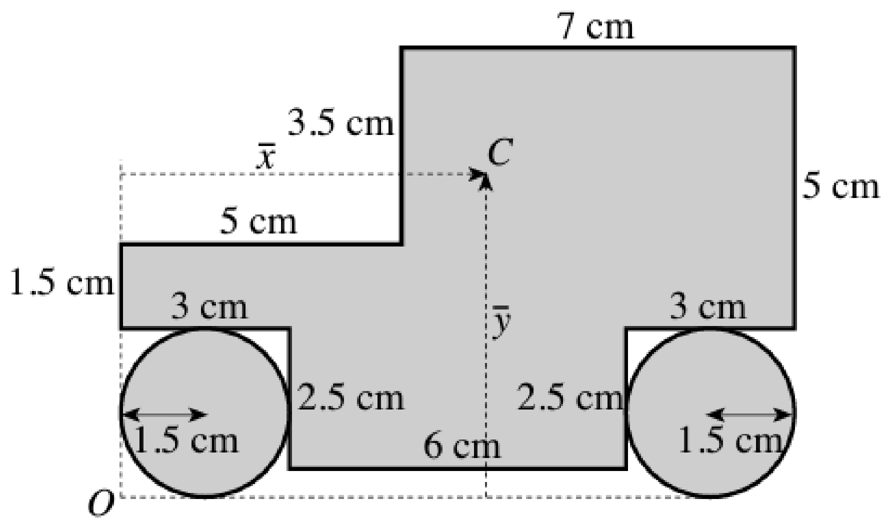
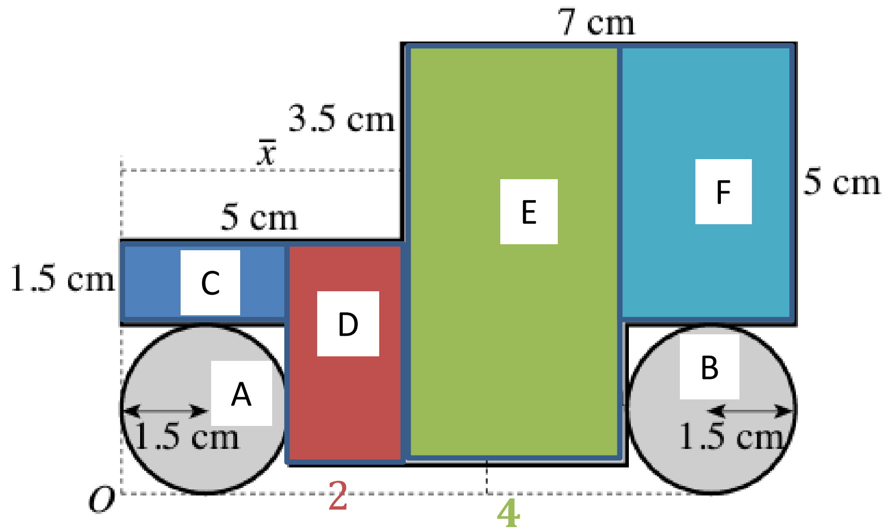
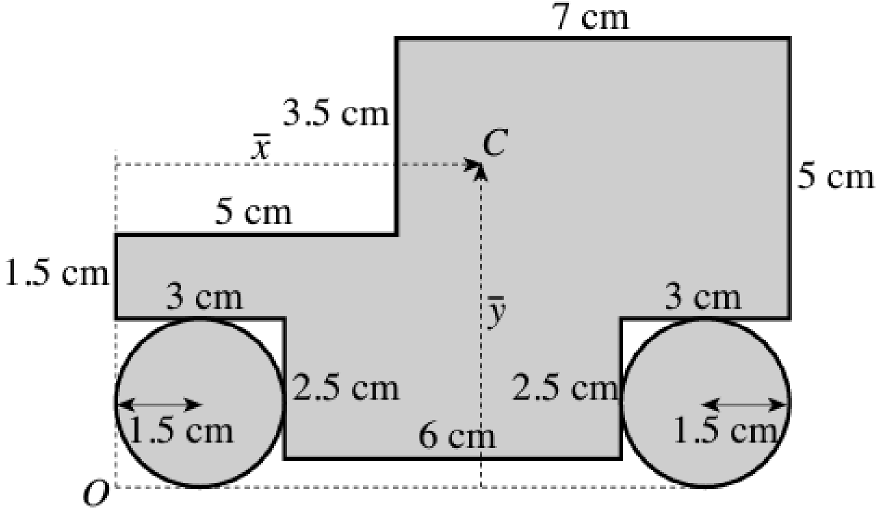
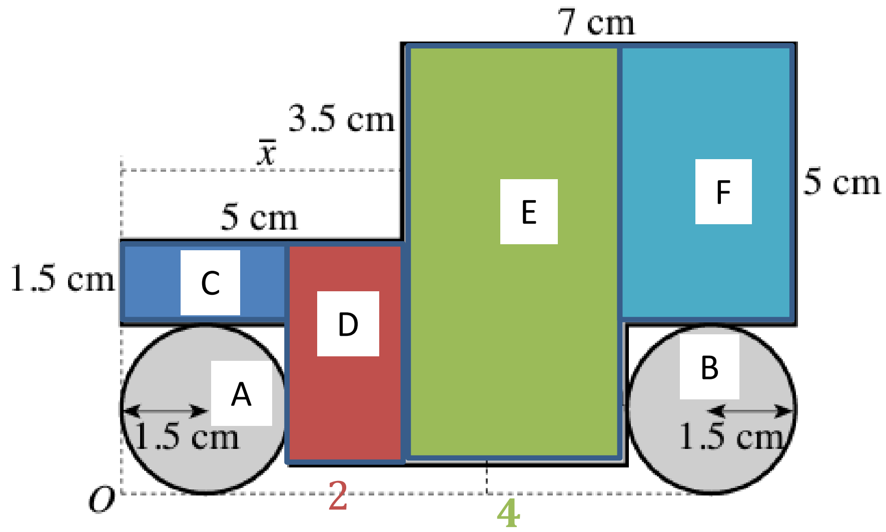




	A	B	C	D	E	F	
m	7.0686	7.0686	4.5	8	30	15	71.64
x	1.5	10.5	1.5	4	7	10.5	
y	1.5	1.5	3.75	2.5	4.25	5.5	

16 A sheet of metal has uniform thickness and uniform density. The shape shown in the diagram is cut from the sheet of metal. Find the coordinates $(\bar{x}, \bar{y})$ of the centre of mass C of the shape.



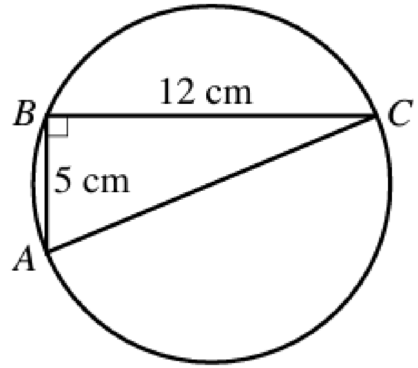


	A	B	C	D	E	F	
m	7.0686	7.0686	4.5	8	30	15	71.64
x	1.5	10.5	1.5	4	7	10.5	$\bar{x} = 6.86$
y	1.5	1.5	3.75	2.5	4.25	5.5	$\bar{y} = 3.74$

The structure shown in the diagram, consisting of a right-angled triangle inscribed in a circle, is made of uniform wire. Calculate the distance of the centre of mass of the structure from

(a) AB,

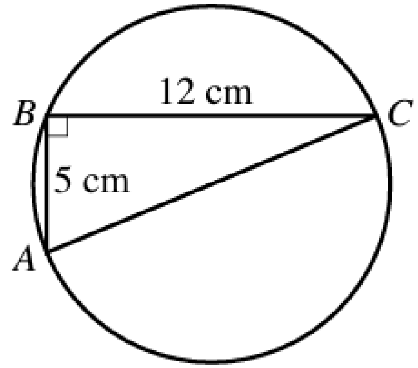
(b) BC.



	Circle	$\overline{BA}$	$\overline{AC}$	C	
Mass					
x					$\bar{x}$
y					$\bar{y}$

The structure shown in the diagram, consisting of a right-angled triangle inscribed in a circle, is made of uniform wire. Calculate the distance of the centre of mass of the structure from

(a) AB, (b) BC.

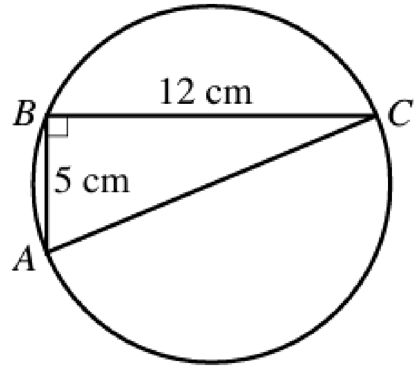


	Circle	$\overline{BA}$	$\overline{AC}$	C	
Mass	$2\pi(6.5)k$				
x	6				$\bar{x}$
y	2.5				$\bar{y}$

The structure shown in the diagram, consisting of a right-angled triangle inscribed in a circle, is made of uniform wire. Calculate the distance of the centre of mass of the structure from

(a) AB,

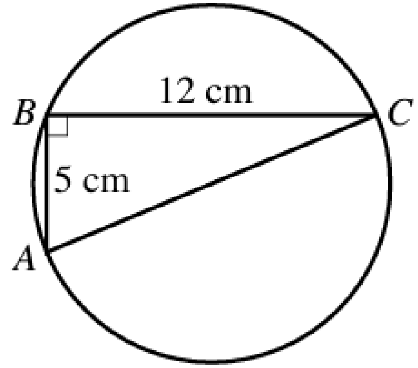
(b) BC.



	Circle	$\overline{BA}$	$\overline{AC}$	C	
Mass	$2\pi(6.5)k$	$5k$			
x	6	0			$\bar{x}$
y	2.5	2.5			$\bar{y}$

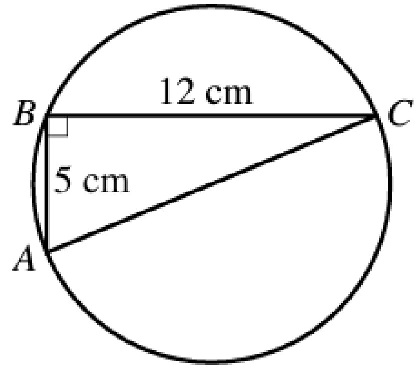
The structure shown in the diagram, consisting of a right-angled triangle inscribed in a circle, is made of uniform wire. Calculate the distance of the centre of mass of the structure from

(a) AB, (b) BC.



	Circle	$\overline{BA}$	$\overline{AC}$	C	
Mass	$2\pi(6.5)k$	$5k$	$13k$		
x	6	0	6		$\bar{x}$
y	2.5	2.5	2.5		$\bar{y}$

The structure shown in the diagram, consisting of a right-angled triangle inscribed in a circle, is made of uniform wire. Calculate the distance of the centre of mass of the structure from

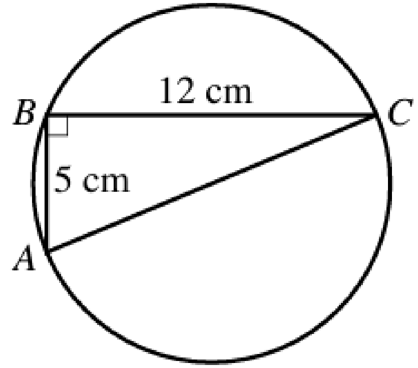


(a) AB, (b) BC.

	Circle	$\overline{BA}$	$\overline{AC}$	C	
Mass	$2\pi(6.5)k$	$5k$	$13k$	$12k$	
x	6	0	6	6	$\bar{x}$
y	2.5	2.5	2.5	0	$\bar{y}$

The structure shown in the diagram, consisting of a right-angled triangle inscribed in a circle, is made of uniform wire. Calculate the distance of the centre of mass of the structure from

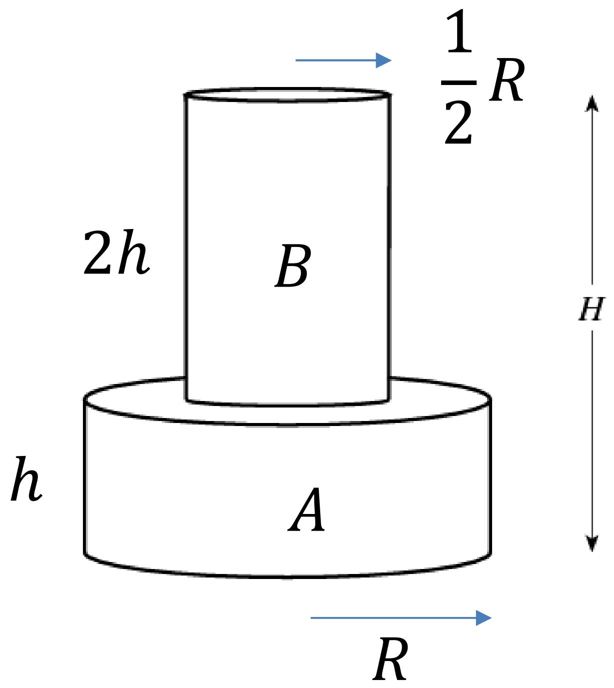
(a) AB, (b) BC.



	Circle	$\overline{BA}$	$\overline{AC}$	C	
Mass	$2\pi(6.5)k$	$5k$	$13k$	$12k$	$70.84k$
x	6	0	6	6	$\bar{x}$
y	2.5	2.5	2.5	0	$\bar{y}$

$\bar{x} = 5.58$ cm $\bar{y} = 2.08$ cm

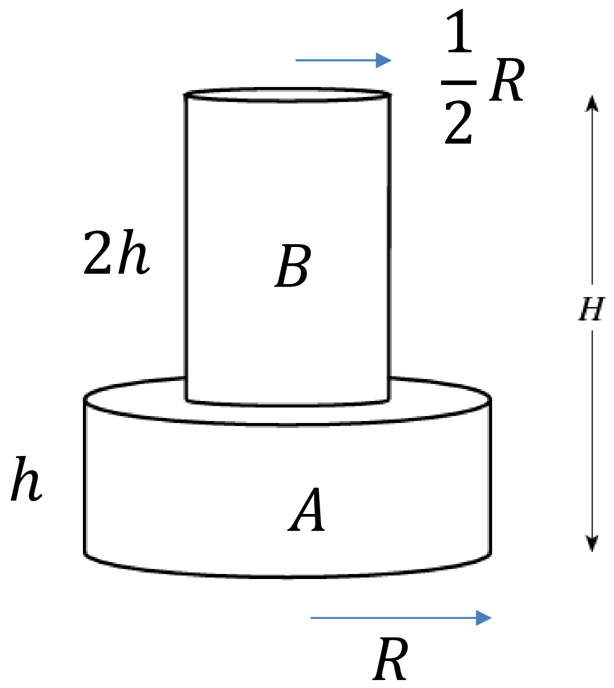
Two uniform cylinders, made from the same material, are arranged one on top of the other as shown in the diagram. The radius of the upper cylinder is half that of the lower, and the height of the upper cylinder is twice that of the lower. The overall height of the structure is H. Find, in terms of H, the distance of the centre of mass from the base.



$$H = 3h \quad \Rightarrow h = 1/3H$$

	A	B	
Mass			
y			

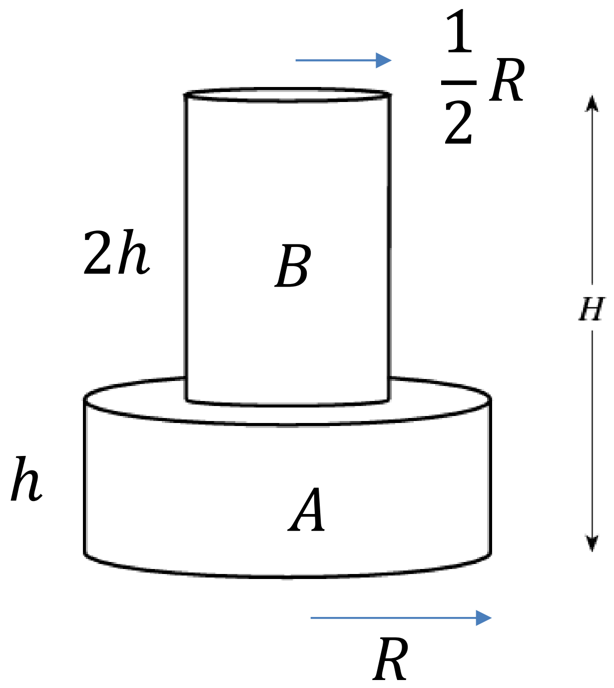
Two uniform cylinders, made from the same material, are arranged one on top of the other as shown in the diagram. The radius of the upper cylinder is half that of the lower, and the height of the upper cylinder is twice that of the lower. The overall height of the structure is H. Find, in terms of H, the distance of the centre of mass from the base.



$$H = 3h \quad \Rightarrow h = 1/3H$$

	A	B	
Mass	$\pi R^2 \frac{1}{3} Hk$	$\pi \frac{1}{4} R^2 \frac{2}{3} Hk$	$\frac{1}{2} \pi R^2 Hk$
y	$\frac{1}{6} H$	$\frac{1}{3} H + \frac{1}{3} H$	

Two uniform cylinders, made from the same material, are arranged one on top of the other as shown in the diagram. The radius of the upper cylinder is half that of the lower, and the height of the upper cylinder is twice that of the lower. The overall height of the structure is H. Find, in terms of H, the distance of the centre of mass from the base.



$$H = 3h \quad \Rightarrow h = 1/3H$$

	A	B	
Mass	$\frac{1}{3}\pi R^2 Hk$	$\frac{1}{6}\pi R^2 Hk$	$\frac{1}{2}\pi R^2 Hk$
y	$\frac{1}{6}H$	$\frac{2}{3}H$	

	A	B	
Mass	$\frac{1}{3}\pi R^2 Hk$	$\frac{1}{6}\pi R^2 Hk$	$\frac{1}{2}\pi R^2 Hk$
y	$\frac{1}{6}H$	$\frac{2}{3}H$	

$$\bar{y} = \frac{\frac{1}{3}\pi R^2 Hk\left(\frac{1}{6}H\right) + \frac{1}{6}\pi R^2 Hk\left(\frac{2}{3}H\right)}{\frac{1}{2}\pi R^2 Hk}$$

$$\bar{y} = \frac{1}{3}H$$

3.3 Hanging and balancing

Example 3.1.1

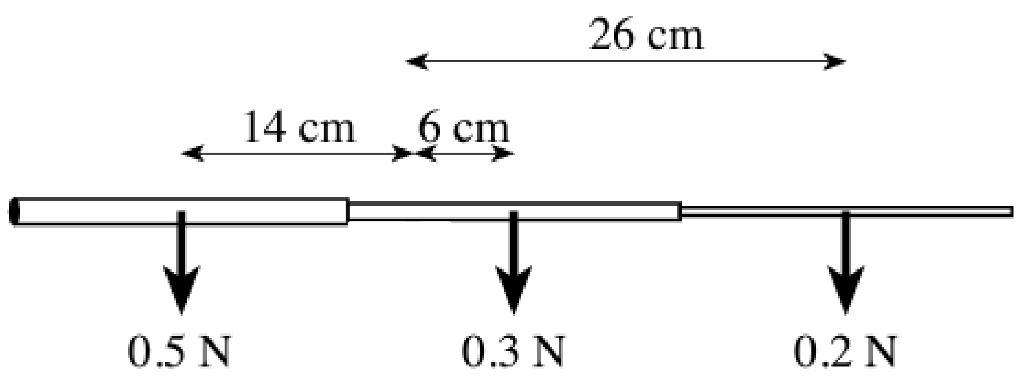
A portable radio has a telescopic aerial, consisting of three parts of masses 0.05 kg, 0.03 kg and 0.02 kg, each of length 20 cm. The first part is hinged to the radio at one end, and the other two parts slide inside it. All three parts are uniform. Find the distance of the centre of mass from the hinge when

(a) the aerial is closed up,

(b) the second and third parts are pulled out together to make an aerial of length 40 cm,

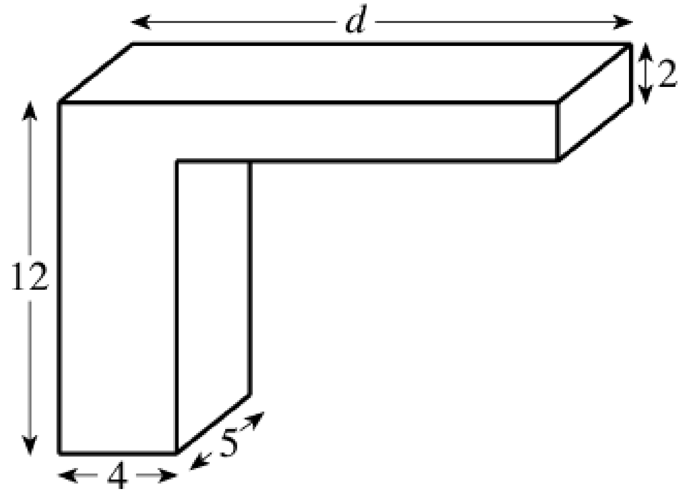
(c) the aerial is fully extended.

Take the aerial in Example 3.1.1 and extend it. Suppose that it could be unhooked from the hinge and balanced on your finger. Where along the aerial would it balance?



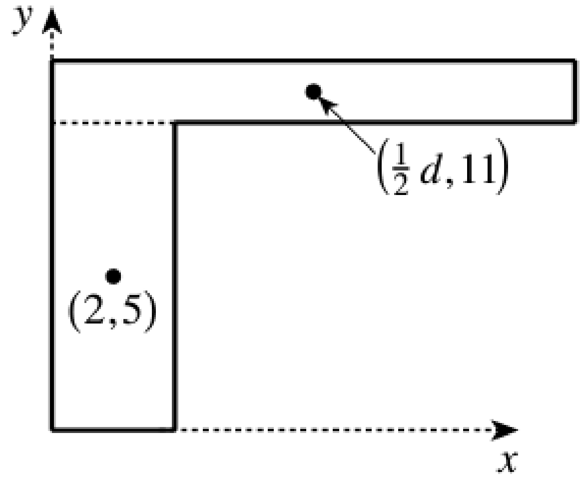
Example 3.3.1

Fig. 3.11 shows a design for a piece of table sculpture which is to be carved out of a single uniform piece of marble. The dimensions are in centimetres. What is the largest possible value for the length labelled d?

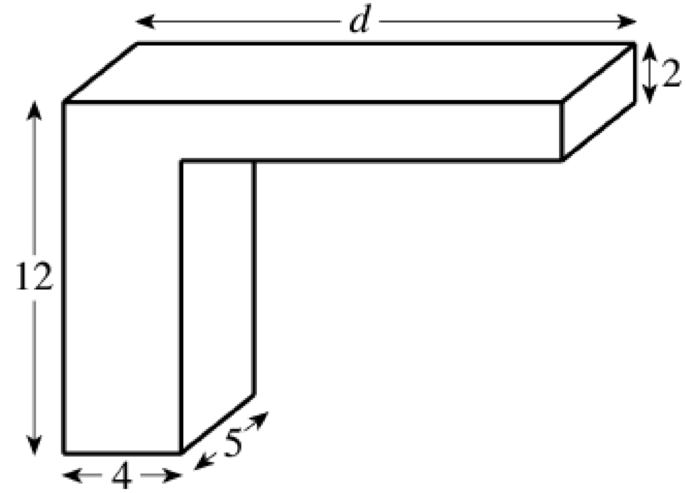


And that the table does not topple over!

Mass (kg)	$200k$	$10dk$	$(200+10d)k$
x-coordinate (cm)	2	$\frac{1}{2}d$	$\bar{x}$
y-coordinate (cm)	5	11	$\bar{y}$



Mass (kg)	$200k$	$10dk$	$(200+10d)k$
x-coordinate (cm)	2	$\frac{1}{2}d$	$\bar{x}$
y-coordinate (cm)	5	11	$\bar{y}$



$$\bar{x} = \frac{200k(2) + 10dk(1/2d)}{(200+10d)k}$$

$$\therefore \bar{x} = \frac{80 + d^2}{40 + 2d} \quad \leq 4$$

$$\therefore d^2 - 8d \leq 80$$

$$\therefore d^2 - 8d + 16 \leq 96$$

$$\therefore (d-4)^2 \leq 96$$

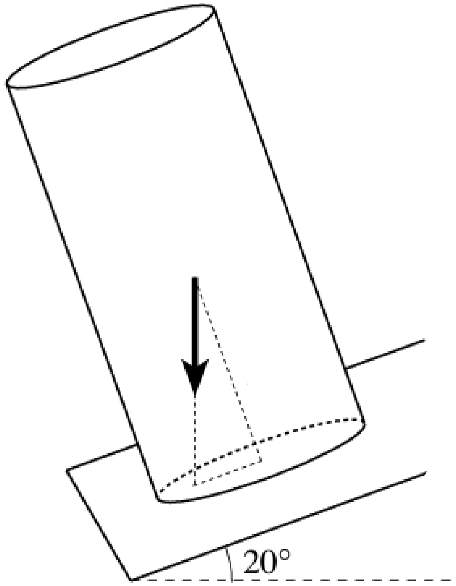
$$\therefore d \leq 13.8$$

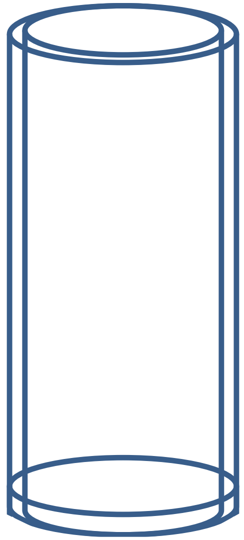
Example 3.3.2

A plastic container has the shape of a cylindrical tube whose height is twice the diameter of its base. The sides and the base have the same thickness. It is placed on a rough surface at $20°$ to the horizontal. Will it topple over if placed

(a) the right way up, (b) upside down?

The container is made of very thin plastic



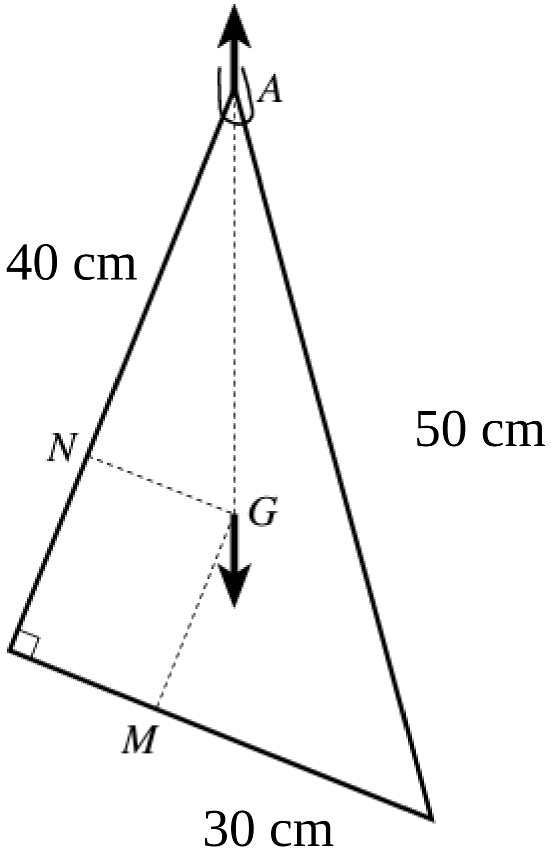


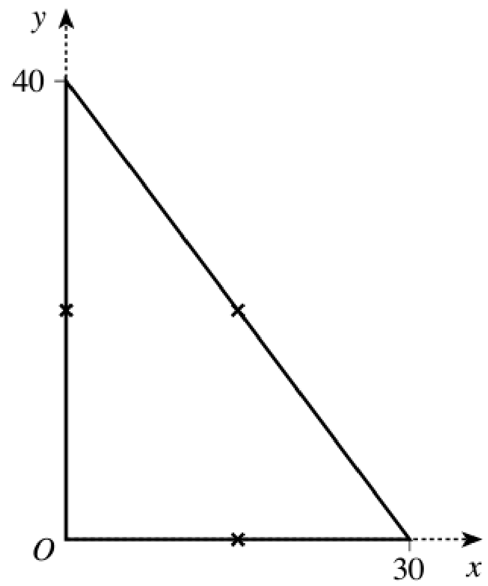
	Base	Cylinder	
Mass	m	$8m$	$9m$
Height above base	0	$2r$	$\bar{y}$

$$\bar{y} = \frac{16mr}{9m} = \tfrac{16}{9}r\,.$$

Example 3.3.3

The triangular wire in Example 3.2.2 hangs from a hook at the sharpest corner. Find the angle which the 40 cm side makes with the vertical.



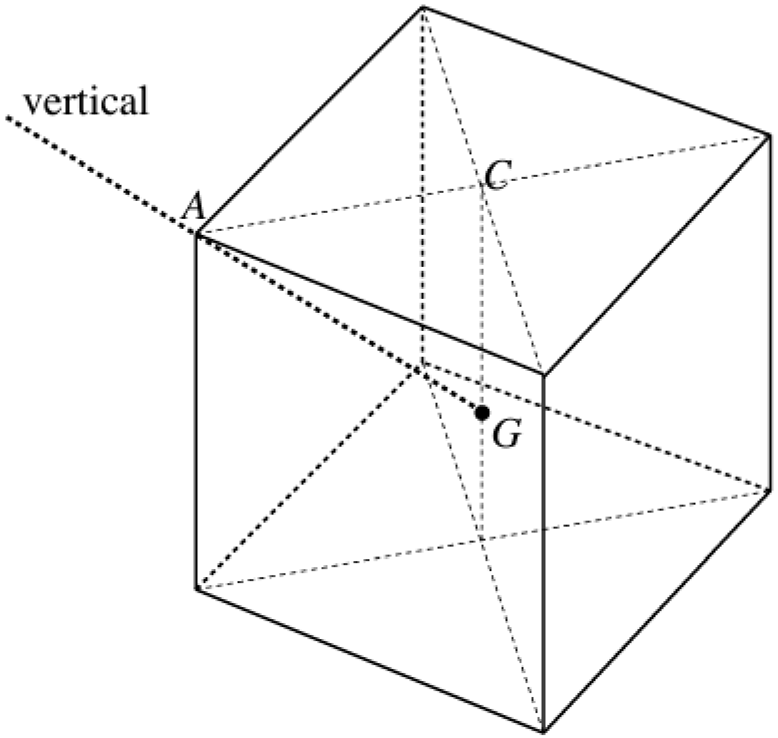


$$\bar{x} = \frac{30k \times 15 + 40k \times 0 + 50k \times 15}{120k} = 10,$$

$$\bar{y} = \frac{30k \times 0 + 40k \times 20 + 50k \times 20}{120k} = 15.$$

Example 3.3.4

An open box has five square faces of side 10 cm, each having mass m kg. Initially it stands with its base on the floor. A string is then fixed to one of the upper corners, and the box is lifted by the string until it is clear of the floor. Find the angle now made with the vertical by the edges which were originally vertical.



Self study