| **Self study** | **Homework (to be handed in)** |
|---|---|
| Examples 4.1.1, 2, 3, 4<br>Examples 4.2.1, 4.2.2 | Exercise 4A: 2, 3<br><br>Exercise 4B: 2, 3 |

# 1. The motion of projectiles

- Understand displacement, velocity and acceleration as vector quantities
- Able to interpret the motion as a combination of the effects of the initial velocity and of gravity
- Know that this implies the independence of horizontal and vertical motion
- Able to use equations of horizontal and vertical motion in calculations about the trajectory of a projectile
- Know and be able to obtain general formulae for the greatest height, time of flight, range on horizontal ground and the equation of the trajectory
- Able to use your knowledge of trigonometry in solving problems

# 1.3 Some general formulae

$$x = u\cos\theta\, t$$

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2$$

$$y = u\sin\theta\, t - \frac{1}{2}gt^2$$

$$y = y_0 + u\sin\theta\, t - \frac{1}{2}gt^2$$

$$\mathbf{v} = \mathbf{u} + \mathbf{g}t$$

$$\dot{x} = u\cos\theta$$

$$\dot{y} = u\sin\theta - gt$$

$$\frac{dx}{dt} \equiv \dot{x} \qquad \frac{dy}{dt} \equiv \dot{y}$$

# 1.3 Some general formulae

Greatest height:

$$h = \frac{u^2 \sin^2 \theta}{2g}$$

Range on horizontal ground:

$$r = \frac{u^2 \sin 2\theta}{g}$$

Maximum range on horizontal ground:

$$r_{\max} = \frac{u^2}{g}$$

Equation of the trajectory:

$$y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$$

For a projectile having initial velocity of magnitude $u$ at an angle $\theta$ to the horizontal, under gravity but neglecting air resistance:

the greatest height reached is $\dfrac{u^2 \sin^2 \theta}{2g}$;

the time to return to its original height is $\dfrac{2u \sin \theta}{g}$;

the range on horizontal ground is $\dfrac{u^2 \sin 2\theta}{g}$;

the maximum range on horizontal ground is $\dfrac{u^2}{g}$;

the equation of the trajectory is

$$y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}, \quad \text{or} \quad y = x \tan \theta - \frac{gx^2 \sec^2 \theta}{2u^2}.$$

# 2. Moments

- Understand the model of a rigid object
- Understand that a rigid object has a centre of mass
- Know that the turning effect of a force is measured by its moment
- Able to calculate the moments of forces
- Able to solve problems about the equilibrium of rigid objects by taking moments about chosen points

# 2.2 Centres of mass

An important property of rigid objects is that they have a point at which, for many purposes, the whole mass may be supposed to be concentrated. This point is called the **centre of mass** of the object.

If an object is made of the same material with the same density all the way through, it is said to be **uniform**. For any uniform rigid object with a centre of geometrical symmetry, the centre of mass is at the centre of symmetry.

# 2.3 The moment of a force

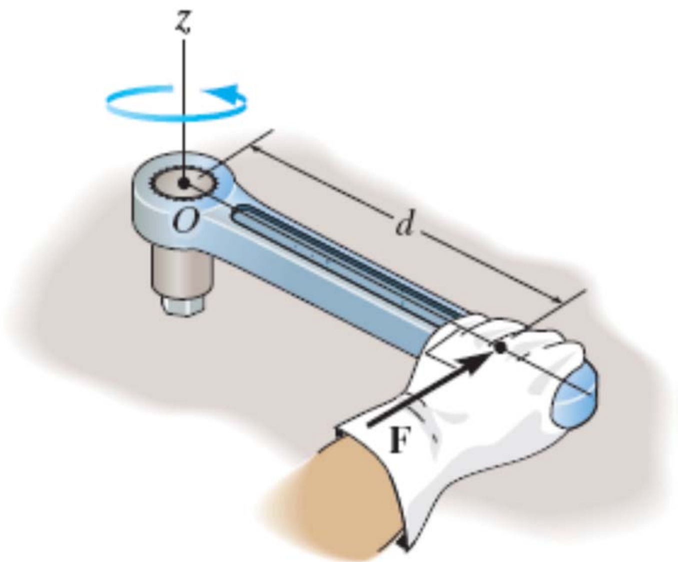
The **moment** of a force about a point is calculated as the product of the magnitude of the force and the distance of its line of action from the point.

$$\sum \boldsymbol{F} = m\boldsymbol{a} \quad (\boldsymbol{a} \to \mathbf{0})$$

$$\sum \boldsymbol{F} = \mathbf{0}$$

$$\circlearrowleft + \sum M_O = 0$$

$$M_O = Fd \quad \text{[N.m]}$$



# 2.4 Forces from supports

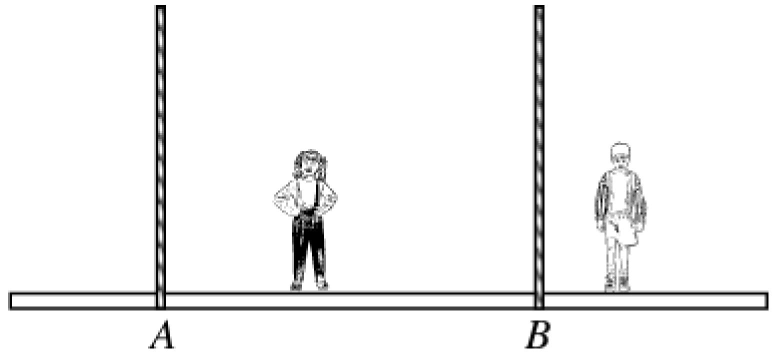
**The principle of moments** If a rigid object is in equilibrium, the sum of the anticlockwise moments about any point must equal the sum of the clockwise moments.

**The principle of moments**: If a rigid object is in equilibrium, the sum of the moments about any point must be equal to zero.

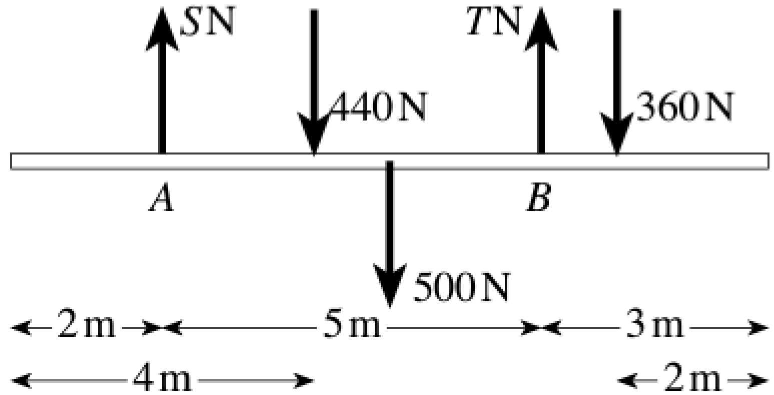
$$\circlearrowleft{+} \quad \sum M_O = 0$$

**Example 2.4.1**

A horizontal uniform plank of length 10 m and mass 50 kg is supported by two vertical ropes, attached at $A$, 2 m from the left end, and at $B$, 3 m from the right end. Two children stand on the plank: Lindi (44 kg) 4 m from the left end, and Jakob (36 kg) 2 m from the right end, as shown in Fig. 2.10. Find the tension in the ropes.



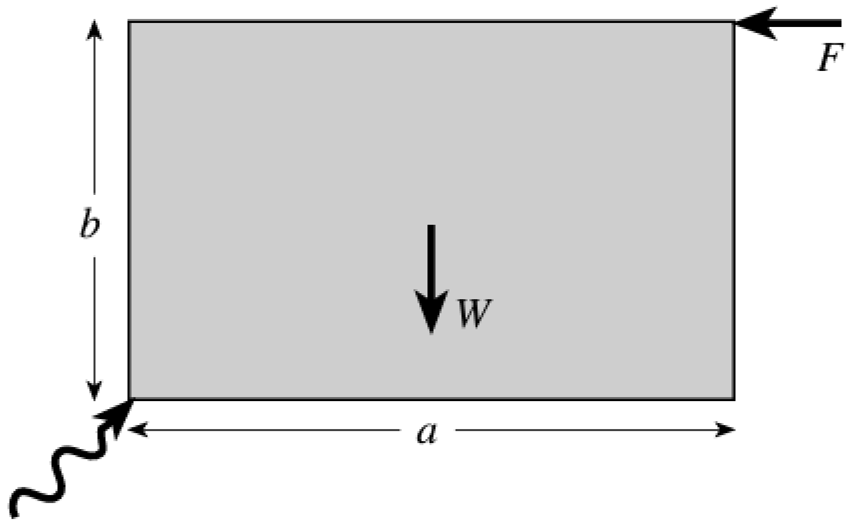

You can remove an unknown or unwanted force from an equation by taking moments about a point on its line of action.



# 2.5 Forces in different directions

**Example 2.5.1**

A uniform rectangular plate of weight $W$ is held in a vertical plane as shown in Fig. 2.14. The plate has width $a$ and height $b$. It is hinged at the lower left corner, and kept in equilibrium by a horizontal force $F$ applied at the upper right corner. Find $F$ in terms of $a$, $b$ and $W$.



# 3. Centre of mass

- Able to find the centre of mass for objects made up of parts whose centres of mass you already know
- Understand how the procedure is justified by the theory of moments
- Know the weighted mean formula for finding centres of mass
- Know how to use the centre of mass to determine equilibrium positions for objects

# 3.2 Two-dimensional objects

If an object is made up of $n$ sections of masses $m_1$, $m_2$, … , $m_n$, each with its centre of mass in a plane and having coordinates $(x_1, y_1)$, $(x_2, y_2)$, … , $(x_n, y_n)$, then the centre of mass has coordinates $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{M}, \qquad \bar{y} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{M}$$

and $M = m_1 + m_2 + \ldots + m_n$.

If the centres of mass of the sections have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, … , $\mathbf{r}_n$, then the centre of mass has position vector $\bar{\mathbf{r}}$, where

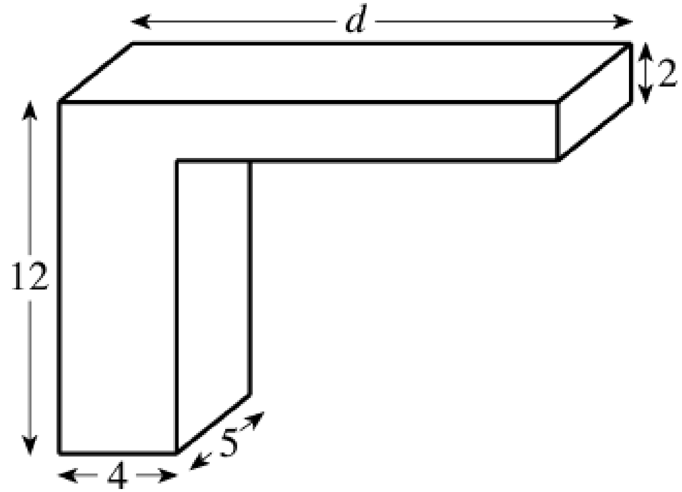
$$\bar{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{M}.$$

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

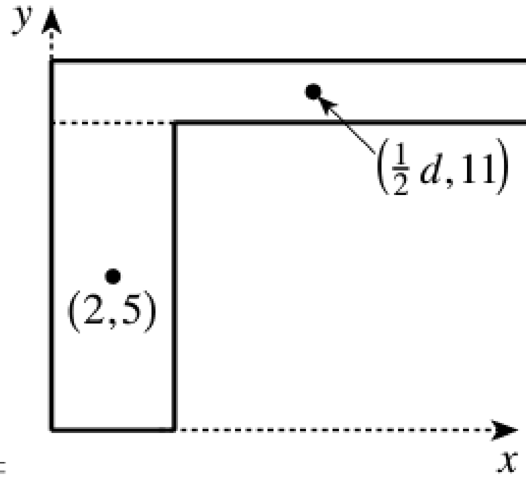
$$\bar{\mathbf{r}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

**Example 3.3.1**

Fig. 3.11 shows a design for a piece of table sculpture which is to be carved out of a single uniform piece of marble. The dimensions are in centimetres. What is the largest possible value for the length labelled $d$?




And that the table does not topple over!




| | | | |
|---|---|---|---|
| Mass (kg) | $200k$ | $10dk$ | $(200+10d)k$ |
| $x$-coordinate (cm) | 2 | $\frac{1}{2}d$ | $\bar{x}$ |
| $y$-coordinate (cm) | 5 | 11 | $\bar{y}$ |

# 4. Rigid objects in equilibrium

- Know and be able to apply general conditions for equilibrium of a rigid object
- Able to determine how equilibrium is broken as a force is increased
- Able to find lines of action of forces needed to maintain equilibrium
- Recognise that some problems are indeterminate and do not have a unique solution

# 4. Rigid objects in equilibrium

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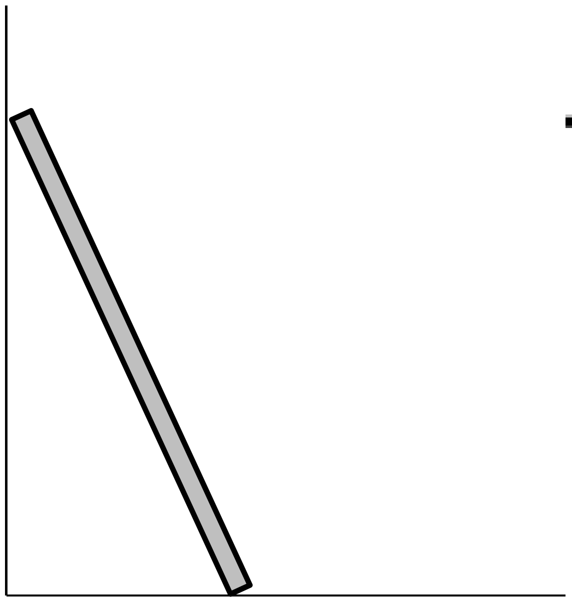
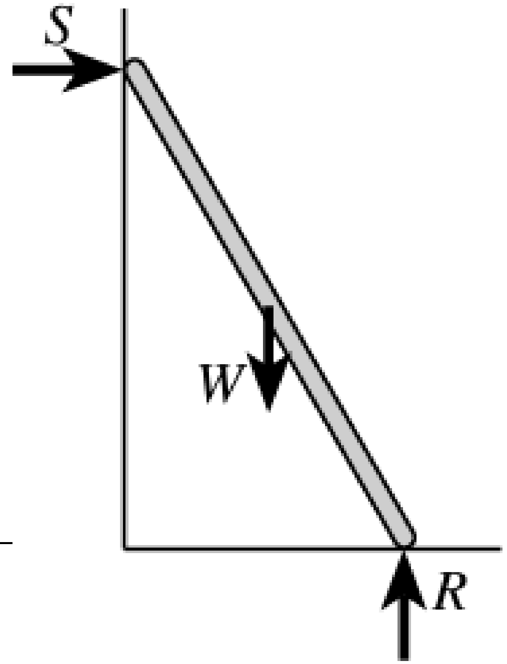
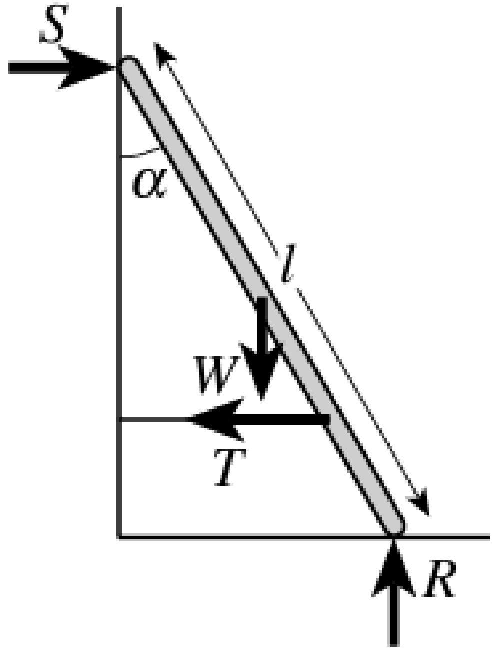
- Able to find lines of action of forces needed to maintain equilibrium
- ~~Recognise that some problems are indeterminate and do not have a unique solution~~

# 4.1 Equilibrium equations



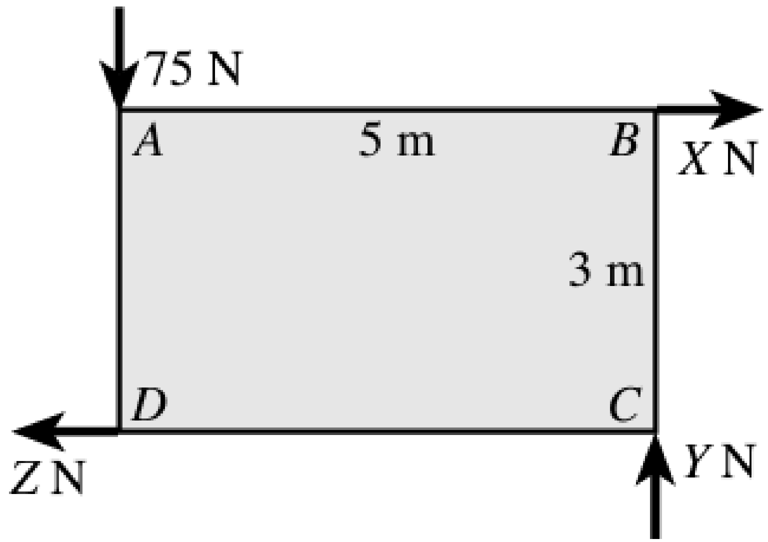



If the surfaces are smooth, can the ladder rest in equilibrium? Explain!

**Example 4.1.1**

A rectangular stone slab $ABCD$ has edges $AB = 5$ m, $BC = 3$ m. It is lying horizontally, and being manoeuvred into position by horizontal forces at $A$, $B$, $C$ and $D$ directed along the edges, as shown in Fig. 4.3. The force at $A$ has magnitude 75 N, and the forces at $B$, $C$ and $D$ are $X$, $Y$ and $Z$ newtons respectively. Calculate the values of $X$, $Y$ and $Z$ needed to keep the slab in equilibrium.




$$\uparrow)\ \sum F_y = 0$$

$$\therefore -75 + Y = 0$$

$$\therefore Y = 75 \text{ N}$$

$$\rightarrow)\ \sum F_x = 0$$

$$\therefore X - Z = 0$$

$$\therefore X = Z$$

$$\circlearrowright)\ \sum M_D = 0$$

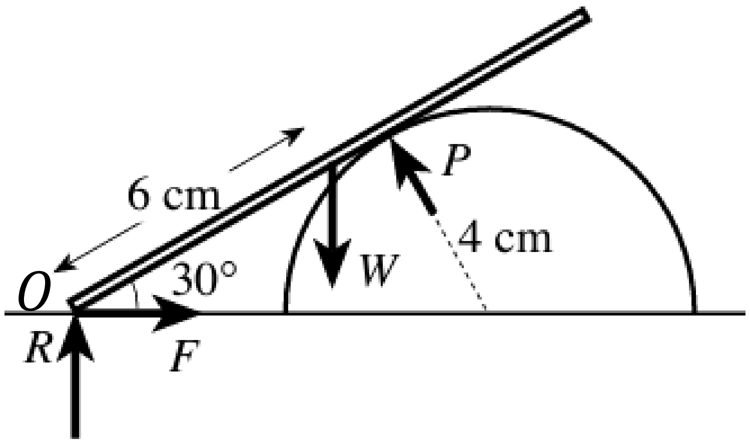
$$\therefore X(3) - 75(5) = 0$$

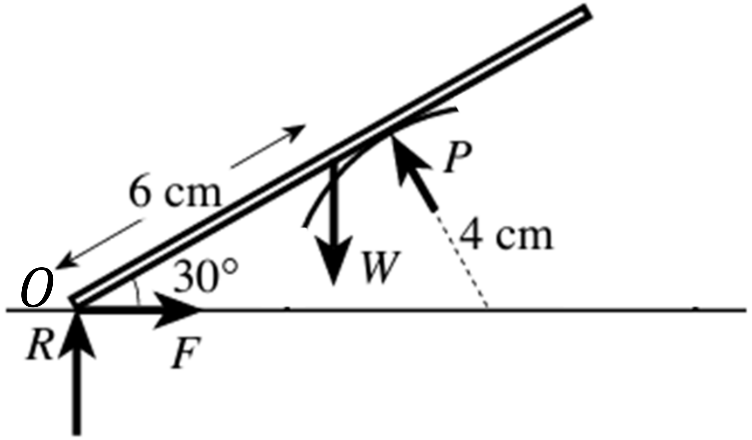
$$\therefore X = 125 \text{ N}$$

$$\therefore Z = 125 \text{ N}$$

**Example 4.1.2**

A student has on her desk a glass paperweight in the shape of a hemisphere of radius 4 cm. She rests a uniform pen 12 cm long against it, at $30°$ to the horizontal, with its lower end on the desk, as shown in Fig. 4.4. The paperweight does not move. If the contact between the pen and the paperweight is smooth, how large must the coefficient of friction between the pen and the desk be to maintain equilibrium?

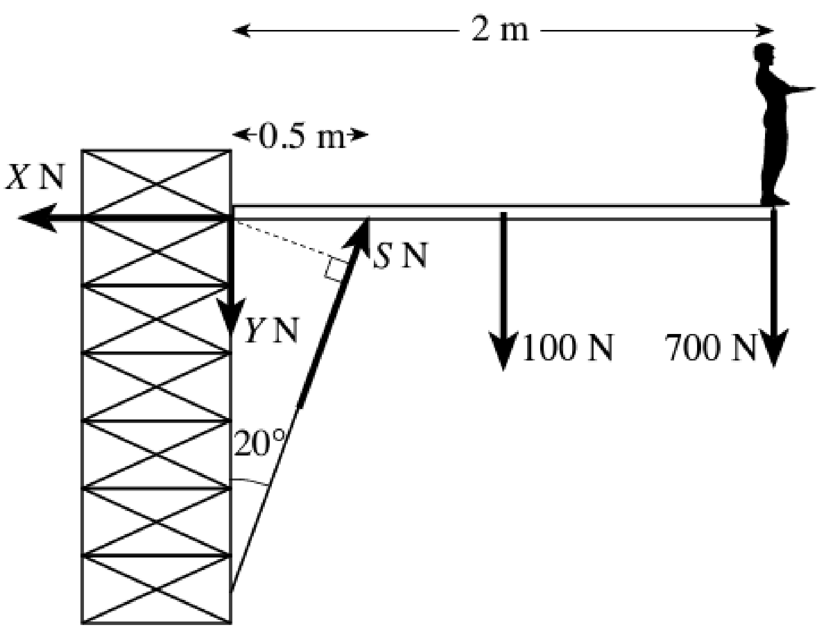


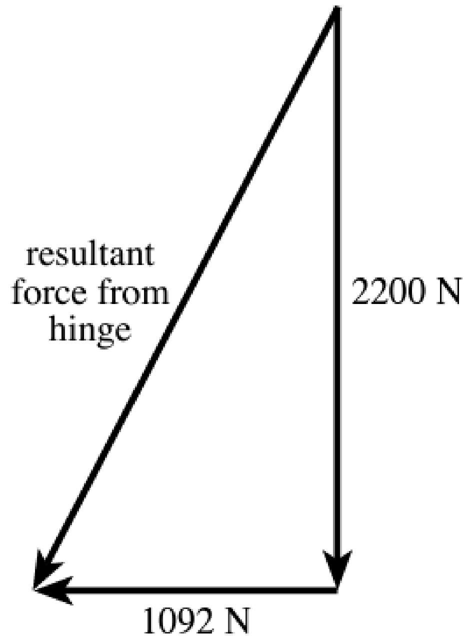




Self study

**Example 4.1.3**

A high diving board is a uniform horizontal plank of length 2 metres and weight 100 N. It is hinged to the step tower at one end, and supported at 0.5 metres from that end by a strut which produces a thrust at 20° to the vertical. Find the magnitude of the thrust and the force from the hinge when a diver of weight 700 N stands at the other end of the board.

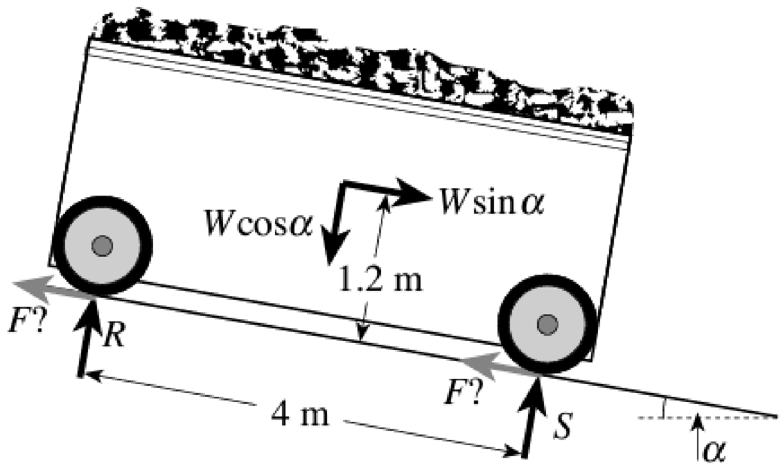



Self study

**Example 4.1.4**

A quarry truck has wheels 4 metres apart. When fully loaded with stone, the centre of mass is 1.2 metres from the rails and midway between the wheels. A brake can be used to lock one of the pairs of wheels when the truck is on a slope. Is it better for the brake to be on the lower or the upper wheels? If the coefficient of friction between the wheels and the rails is 0.4, what is the steepest slope on which the truck can stand when fully loaded?



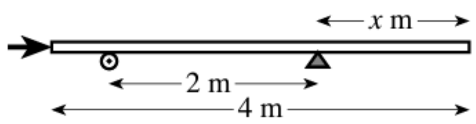

Self study

The equations for the equilibrium of a rigid object acted on by by coplanar forces can be obtained by

- taking moments about one point and resolving in two different directions; or
- taking moments about two points and resolving in a direction which is not perpendicular to the line joining them; or
- taking moments about three points which are not collinear.

# Exercise 4A pg. 63

**1** A uniform metal bar, of length 4 metres and weight 2000 newtons, is being pushed horizontally from one end across two supports 2 metres apart. The support closer to the pushing force is a light rail which can rotate smoothly about a horizontal axis. The other support is fixed, and the coefficient of friction at this support is 0.6. Calculate the force necessary to push the bar at constant speed when $x$ metres of its length projects beyond the fixed support.




$\uparrow) \sum F_y = 0: \; N_1 + N_2 - W = 0$

$\therefore N_1 + N_2 = 2000$

$\rightarrow) \sum F_x = 0: \; F - F_f = 0$

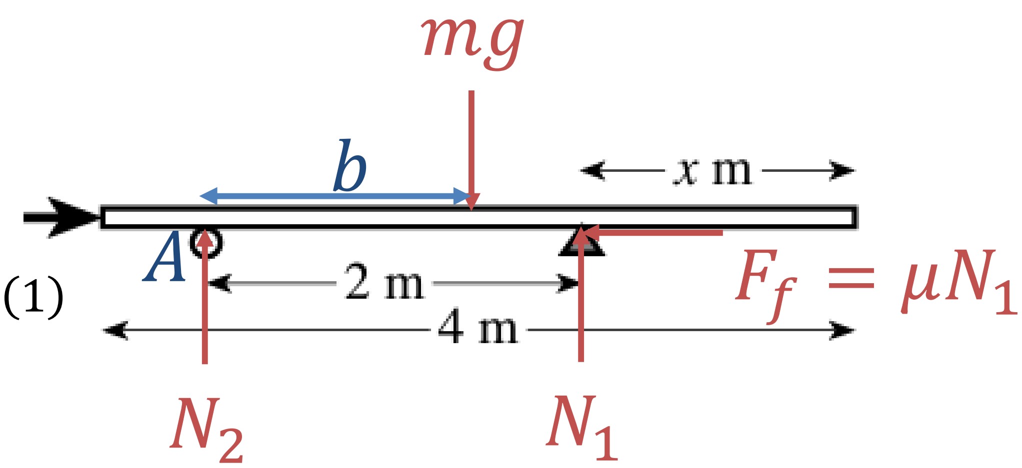
$\therefore F = F_f = \mu N_1 \quad ---(1)$

$\circlearrowleft) \sum M_A = 0: \; W(b) - 2N_1 = 0$

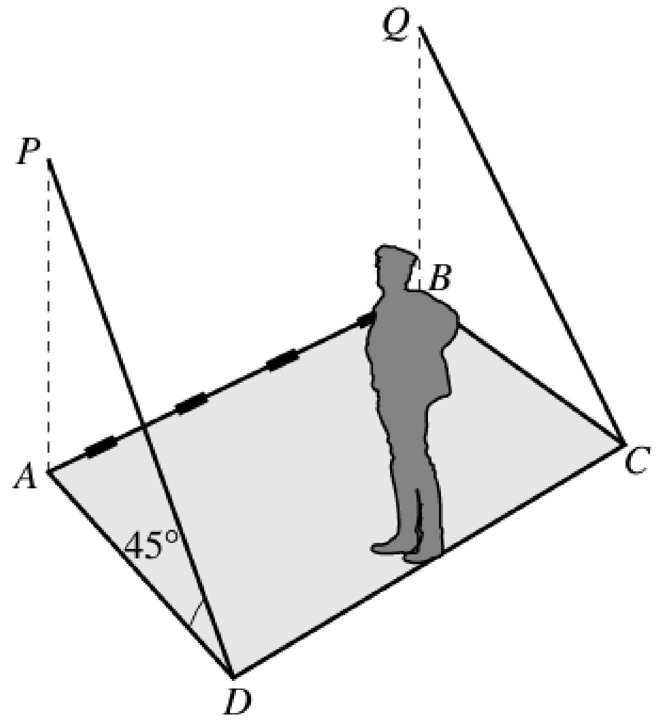
but $b = 2 - a = 2 - (4 - 2 - x) = x$

$W(x) - 2N_1 = 0 \qquad \Rightarrow N_1 = \dfrac{Wx}{2}$

$\therefore F = F_f = \mu \dfrac{Wx}{2} = 0.6 \dfrac{2000x}{2} = 600x$



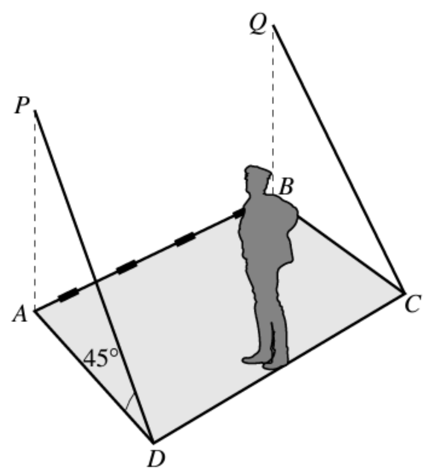
**5** A rectangular platform $ABCD$ of weight $200\text{ N}$ is smoothly hinged, along its edge $AB$, to a vertical wall. The platform is kept horizontal by two parallel chains, inclined at $45°$ to the horizontal, connecting the points $P$ and $Q$ of the wall to the points $D$ and $C$ respectively, as shown in the diagram. $P$ and $Q$ are vertically above $A$ and $B$ respectively. A man of weight $850\text{ N}$ stands on the edge of the platform midway between $D$ and $C$. Find




(a) the tension in each of the chains,

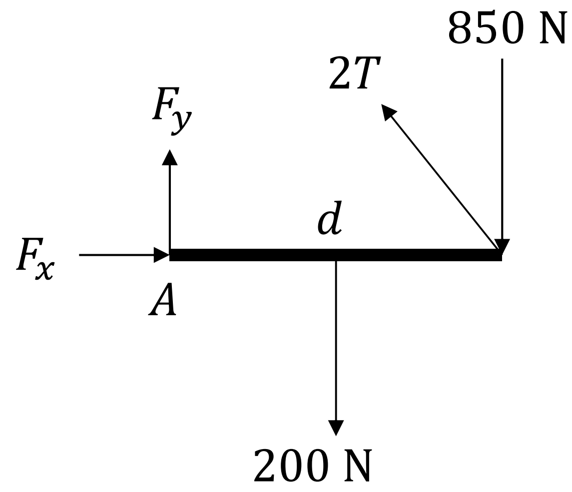
(b) the magnitude of the total force exerted on the hinge by the wall.

**5** A rectangular platform $ABCD$ of weight 200 N is smoothly hinged, along its edge $AB$, to a vertical wall. The platform is kept horizontal by two parallel chains, inclined at 45° to the horizontal, connecting the points $P$ and $Q$ of the wall to the points $D$ and $C$ respectively, as shown in the diagram. $P$ and $Q$ are vertically above $A$ and $B$ respectively. A man of weight 850 N stands on the edge of the platform midway between $D$ and $C$. Find




(a) the tension in each of the chains,

(b) the magnitude of the total force exerted on the hinge by the wall.




$$\circlearrowleft)\ \sum M_A = 0:\ -200\left(\frac{d}{2}\right) - 850(d) + 2T\sin 45^\circ\, d = 0$$

$$\therefore T = 672 \text{ N}$$

$$\uparrow)\ \sum F_y = 0:\ -200 - 850 + 2(672)\sin 45^\circ + F_y = 0$$

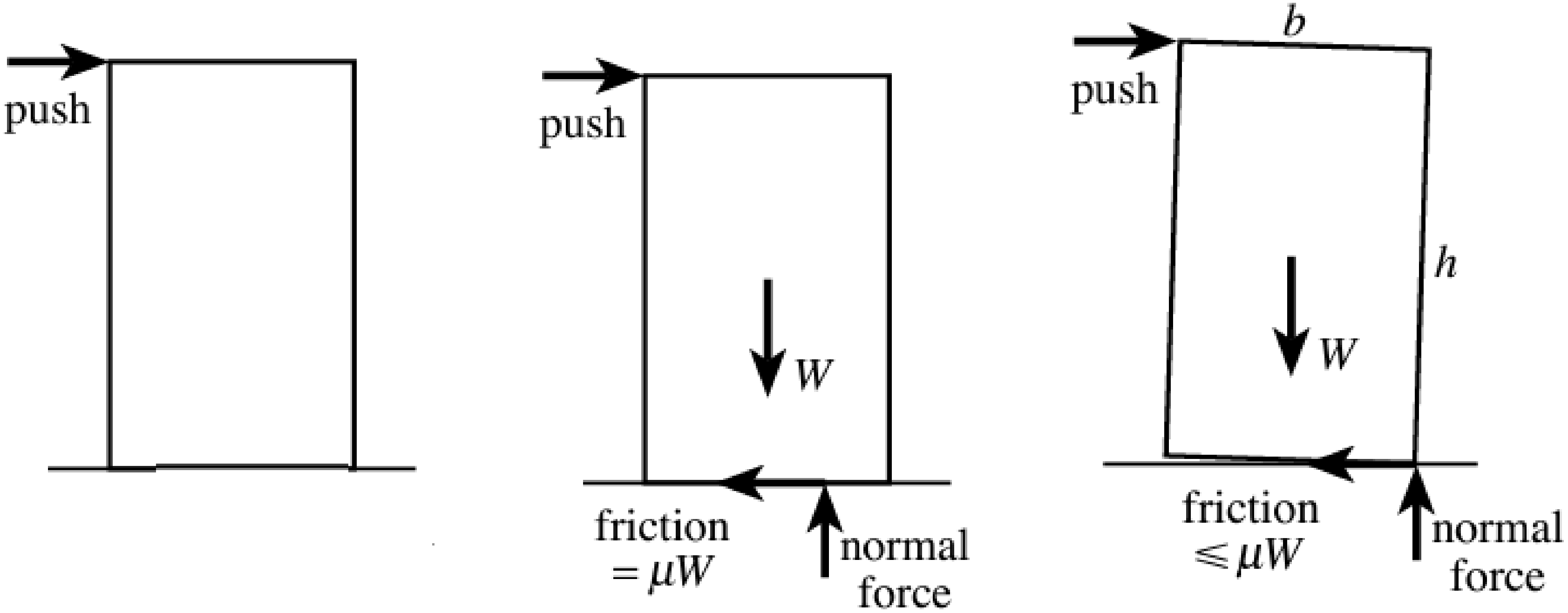
$$\therefore F_y = 99.65 \text{ N}$$

$$\rightarrow)\ \sum F_x = 0:\ F_x - 2(672)\cos 45^\circ = 0$$

$$\therefore F_x = 950.35 \text{ N}$$

$$\therefore F = \sqrt{F_x^2 + F_y^2} = \sqrt{99.65^2 + 950.35^2} = 955.6 \text{ N}$$

# 4.2 Breaking equilibrium by sliding or toppling



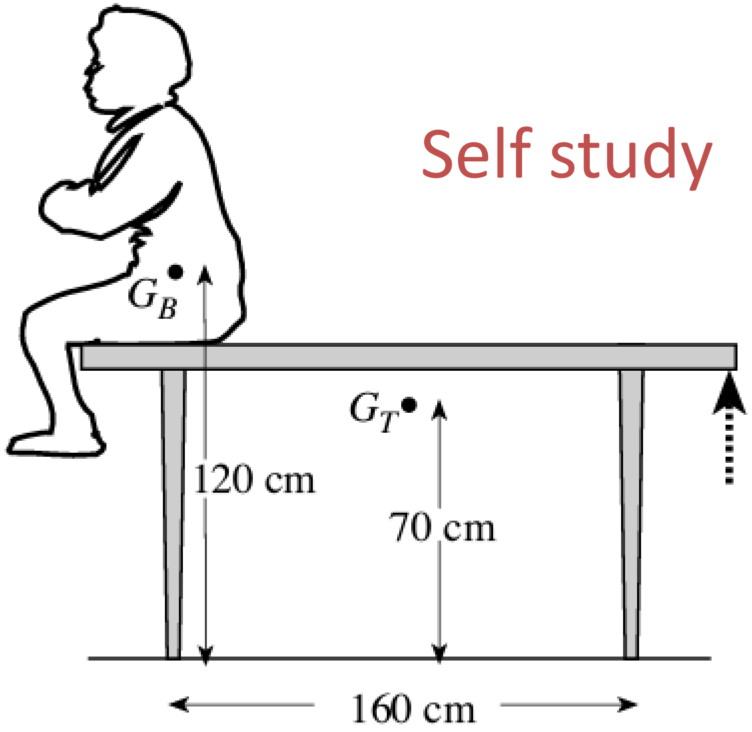

Sliding

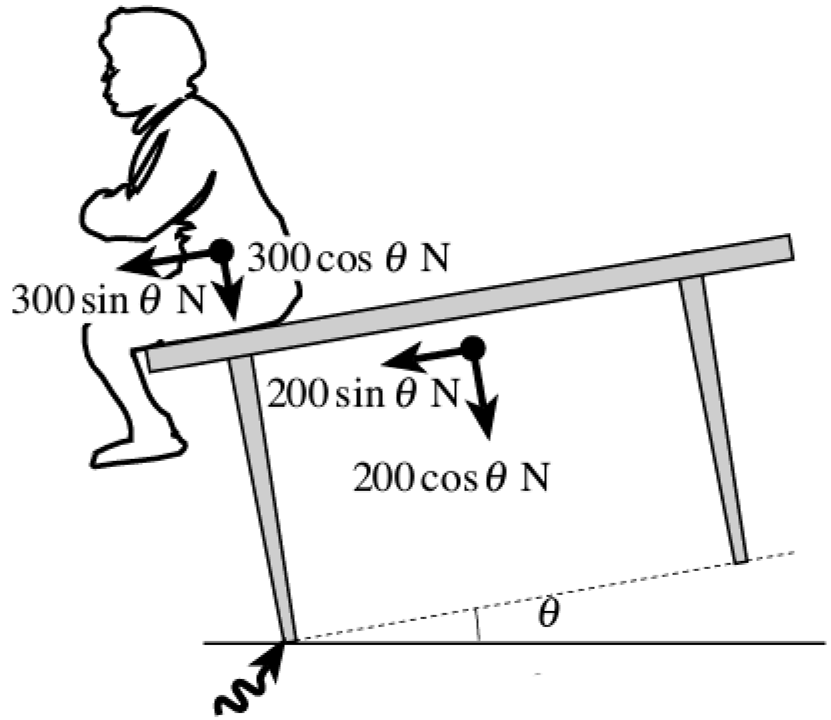
Toppling

**Example 4.2.1**

A small boy of weight 300 N sits on a square table, dangling his legs over the edge. His centre of mass $G_B$ is directly above the line of the legs, 120 cm above the floor. The table has mass 20 kg, and its centre of mass $G_T$ is 70 cm above the floor. The feet of the legs form a square of side 160 cm. The floor is very rough, and the coefficient of friction between the seat of the boy's trousers and the table is 0.3. His sister starts to tilt the table about the legs above which the boy is sitting. Will he slide off the table, or will the boy and the table topple over together?

Self study



3 An Arctic explorer drags a sledge across a horizontal ice-field by means of a rope attached to his body harness. The rope is 5 metres long, and attached to him at a height of 1.4 metres. The sledge is 5 metres long, and the coefficient of friction between the sledge and the ice is 0.2. Where must the centre of mass of the loaded sledge be if the sledge is not to tip up as it is pulled?

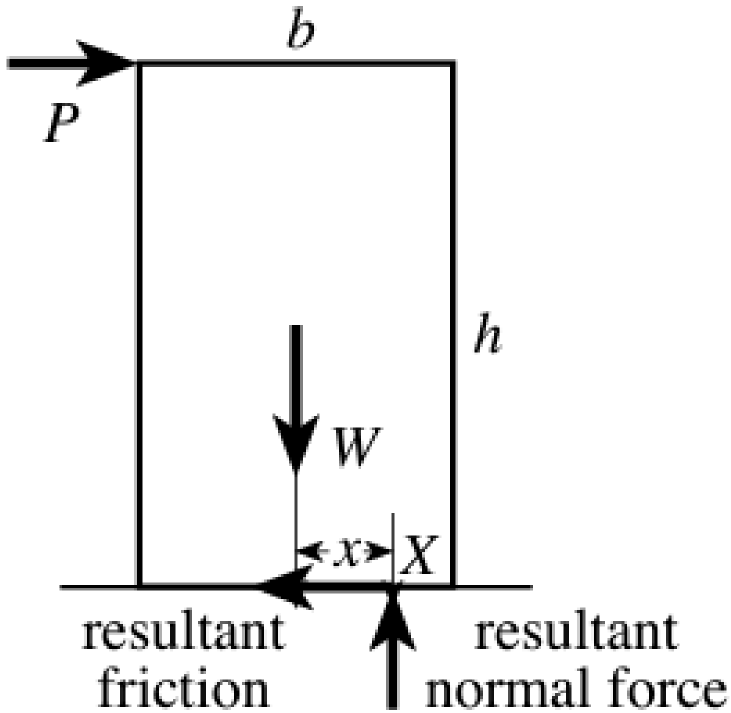
## Example 4.2.2

Self study

The equations for the equilibrium of a rigid object acted on by by coplanar forces can be obtained by

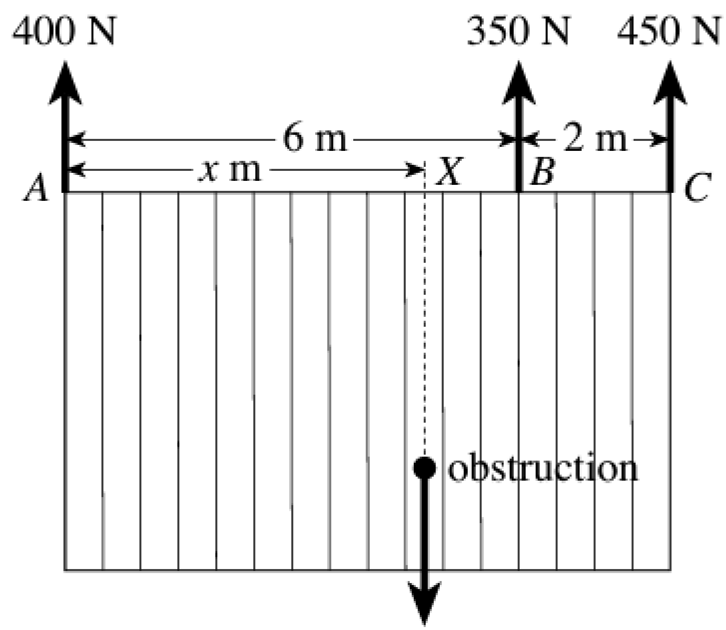
- taking moments about one point and resolving in two different directions; or
- taking moments about two points and resolving in a direction which is not perpendicular to the line joining them; or
- taking moments about three points which are not collinear.

# 4.3 Locating lines of action



**Example 4.3.1**

A raft is stuck on a hidden obstruction in a river-bed (see Fig. 4.15). Ropes are attached to one edge of the raft at $A$, $B$ and $C$, where $AB = 6$ m and $CB = 2$ m. The crew pull on these ropes with forces of 400 N, 350 N and 450 N at right angles to the edge of the raft, but fail to move it. Where is the obstruction?



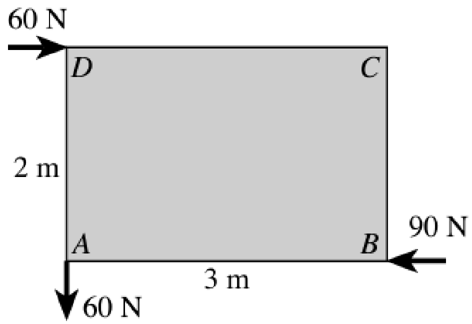

↺) $\sum M_X = 0$:

$$400x - 350(6 - x) - 450(8 - x) = 0$$

$$x = 4.75 \text{ m}$$

**Example 4.3.2**

In a mechanism a rectangular plate $ABCD$ is subjected to forces at $A$, $B$ and $D$ as shown in Fig. 4.16. Where should a fourth force be applied to keep the plate in equilibrium?




$$\rightarrow)\ \sum F_x = 0: \quad 60 - 90 + F_x = 0$$

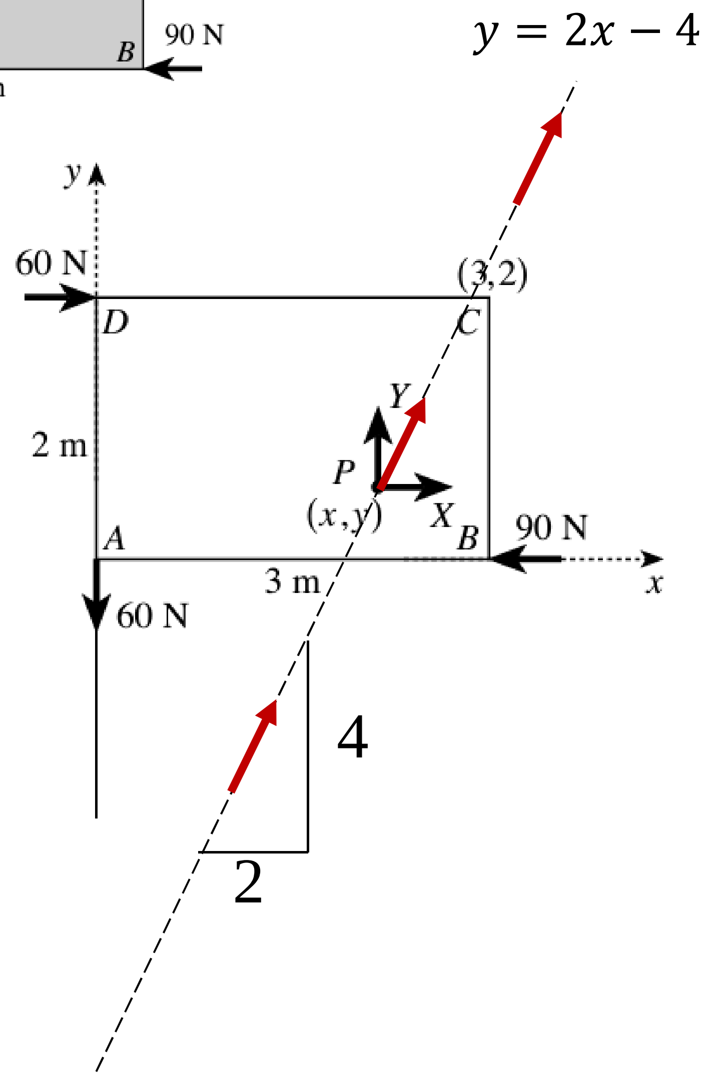
$$F_x = 30\ \text{N}$$

$$\uparrow)\ \sum F_y = 0: \quad F_y - 60 = 0 \quad \therefore F_y = 60\ \text{N}$$

$$\circlearrowleft)\ \sum M_P = 0:$$

$$-60(x) + 60(2 - y) + 90(y) = 0$$

$$y = 2x - 4$$



# Exercise 4B pg. 71

**4** A cylindrical tin has radius 6 cm and height 30 cm, and weighs 100 newtons. The tin stands on a table, 40 cm from the edge, which is smooth and rounded. A string is attached to the point of the top rim of the tin closest to the edge of the table; it passes over the edge and holds a bucket at its other end. Water is poured into the bucket until equilibrium is broken.

(a) How much must the bucket weigh before the tin starts to topple over?

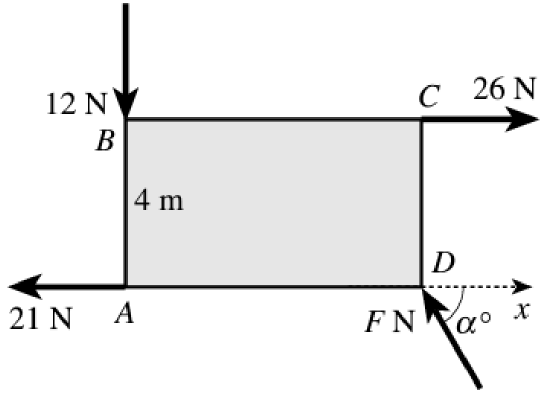
(b) What can be said about the coefficient of friction between the tin and the table if the tin doesn't slide before it begins to topple?

**8** Forces act on the plate $ABCD$ as shown. The distance $AB$ is 4 metres. Given that the plate is in equilibrium, find

(a) $F$,

(b) the angle $\alpha°$,

(c) the distance $AD$.



A uniform straight rod $AB$ has length $l$ and weight $2kW$. The rod is in equilibrium with the end $B$ in contact with a smooth vertical wall and the end $A$ in contact with rough horizontal ground. The rod lies in a vertical plane perpendicular to the wall. The angle between the rod and the ground is $\theta°$. A particle of weight $W$ is attached to the rod at the point whose distance from $A$ is $xl$. The magnitude of the force exerted by the wall on the rod is $\sqrt{3}kW$.

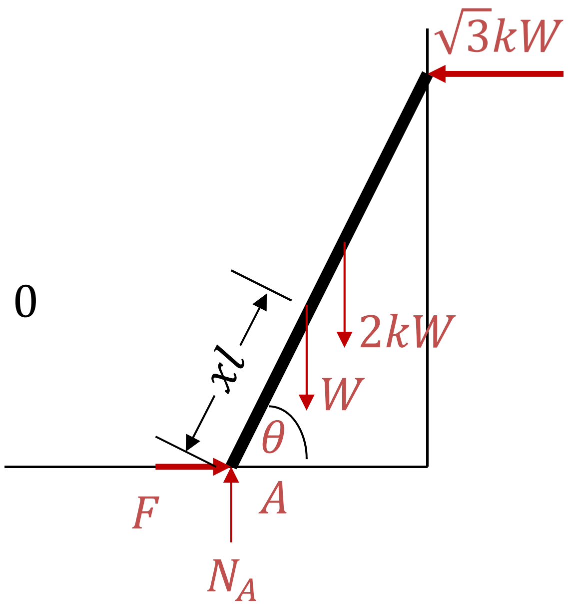
(a) Find, in terms of $k$ and $W$, the horizontal and vertical components of the force exerted by the ground on the rod.

(b) Find $x$ in terms of $k$ and $\theta$.

Deduce that $\theta$ cannot be less than 30. (OCR)

$$\rightarrow)\ \sum F_x = 0: \quad F - \sqrt{3}kW = 0$$

$$F = \sqrt{3}kW$$




$$\uparrow)\ \sum F_y = 0: \quad N_A - W - 2kW = 0$$

$$\therefore N_A = (2k+1)W$$

$$\circlearrowleft)\ \sum M_A = 0: \quad Wxl\cos\theta + \frac{l}{2}\cos\theta\, 2kW - \sqrt{3}kWl\sin\theta = 0$$

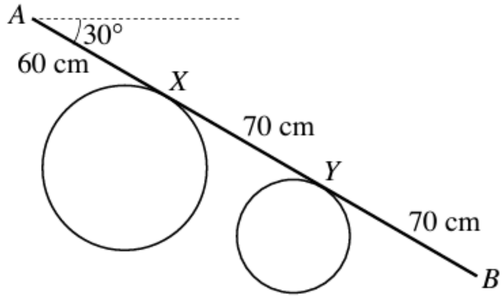
$$x(\cos\theta) = k\left(\sqrt{3}\sin\theta - \cos\theta\right)$$

$$x = k\left(\sqrt{3}\tan\theta - 1\right)$$

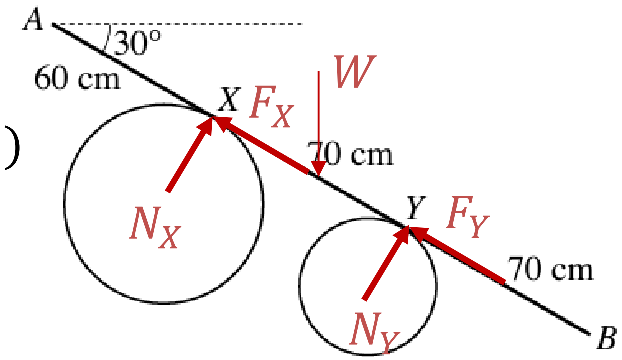
$$\therefore \left(\sqrt{3}\tan\theta - 1\right) \geq 0 \qquad \therefore \tan\theta \geq 1/\sqrt{3}$$

$$\therefore \theta \geq 30°$$

Two circular pipes are fixed with their axes horizontal and parallel to each other. A uniform plank $AB$, of length 2 m and weight 162 N, rests on the pipes, with $A$ higher than $B$, in a vertical plane which is perpendicular to the axes of the pipes (see diagram). The plank touches one of the pipes at $X$ and the other at $Y$, where $AX = 60$ cm, $XY = 70$ cm and $X$ is higher than $Y$. The plank makes an angle of $30°$ with the horizontal. Calculate the normal components of the contact forces on the plank at $X$ and $Y$.




The coefficient of friction has the same value $\mu$ at each contact. Given that the plank is on the point of slipping, find $\mu$. (OCR)




$$\nearrow) \sum F_y = 0: \quad -W\cos 30° + N_X + N_Y = 0$$

$$-162\cos 30° + N_X + N_Y = 0 \quad ----(1)$$

$$\curvearrowleft) \sum M_Y = 0: \quad -N_X(0.7) + 162\cos 30°(0.3) = 0$$

$$\therefore N_X = 60.1 \text{ N}$$

Then from (1): $N_Y = 80.2$ N

$$\searrow) \sum F_x = 0: \quad -F_x - F_Y + W\sin 30° = 0 \quad \therefore -\mu N_x - \mu N_Y + W\sin 30° = 0$$

$$\therefore \mu = \frac{W\sin 30°}{N_X + N_Y} = \frac{162\sin 30°}{60.1 + 80.2} = 0.577$$