Self study	Homework (to be handed in)
Read pp. 79 – 84 Example 5.1.1-5.1.4 Example 5.2.1 Exercise 5A: 1, 3, 5, 9, 13	Exercise 5A: 3, 5, 9, 13

5. Elastic strings and springs

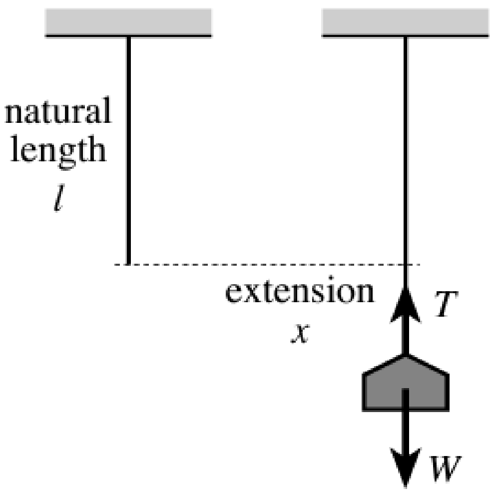
- Able to use Hooke's law as a model relating to the extension of a string, or to the extension and compression of a spring
- Understand the terms 'stiffness' and 'modulus of elasticity'
- Able to derive and use the formula for the work done in stretching a string, or in stretching or compressing a spring
- Understand that elastic forces are conservative, and know how to include elastic potential energy in energy calculations

5.1 The elastic string model

Hooke's law If forces are applied at the ends of a string, rope or cable, then over some range of values the tension is proportional to the extension beyond the natural length.

$T \propto x$

$T = kx$



For an elastic string of natural length l, the tension T and extension x are connected by the equation $T = \frac{\lambda x}{l}$. The constant λ is called the **modulus of elasticity** of the string.

Example 5.1.1

A climber of mass 70 kg hangs from a rope secured at its upper end to a fixed metal ring. The natural length of the rope is 20 metres, and it stretches to 22 metres when supporting the climber. What is the modulus of elasticity of the rope?

> For an elastic string of natural length l, the tension T and extension x are connected by the equation $T = \frac{\lambda x}{l}$. The constant λ is called the **modulus of elasticity** of the string.

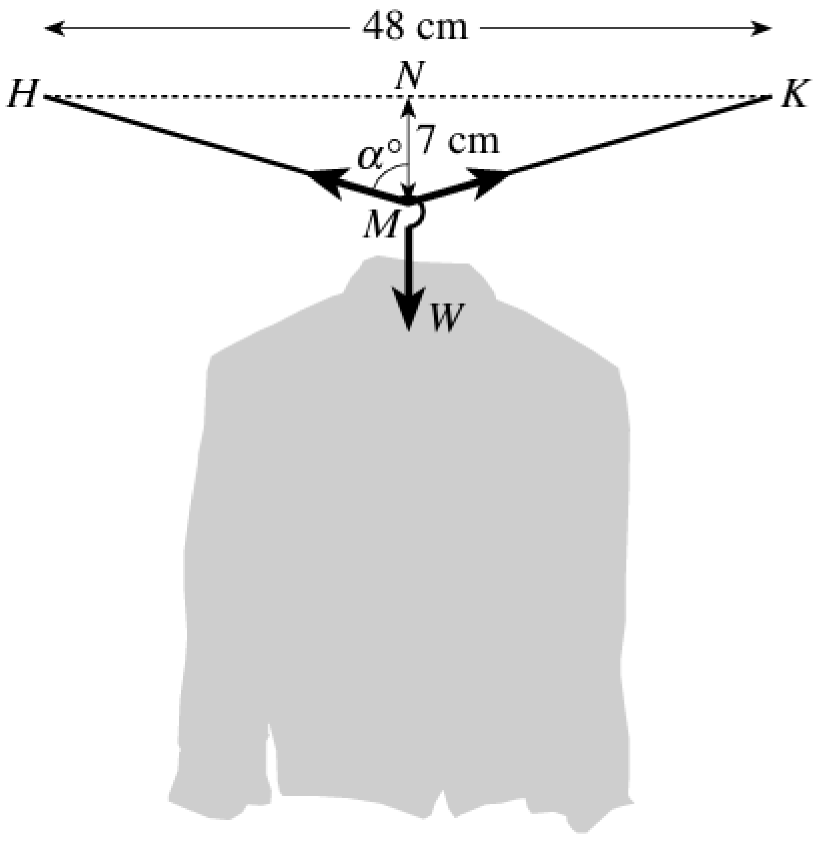
Example 5.1.2

A spider of mass 2 grams hangs from a thread which it is spinning. This has length 7 cm. A typical piece of the web has modulus of elasticity 0.05 newtons. What is the natural length of the thread from which the spider is hanging?

Self study!

Example 5.1.3

A student uses a 40 cm length of curtain wire as a washing line. One end is attached to a hook H on the wall. When he has some laundry to dry, he stretches the wire so that the other end reaches another hook K at the same level 48 cm away. The tension in the wire is then 10 newtons. When the student hangs a wet shirt at the mid-point M of the wire, the wire stretches further so that M is 7 cm below the mid-point N of HK. How much does the wet shirt weigh?



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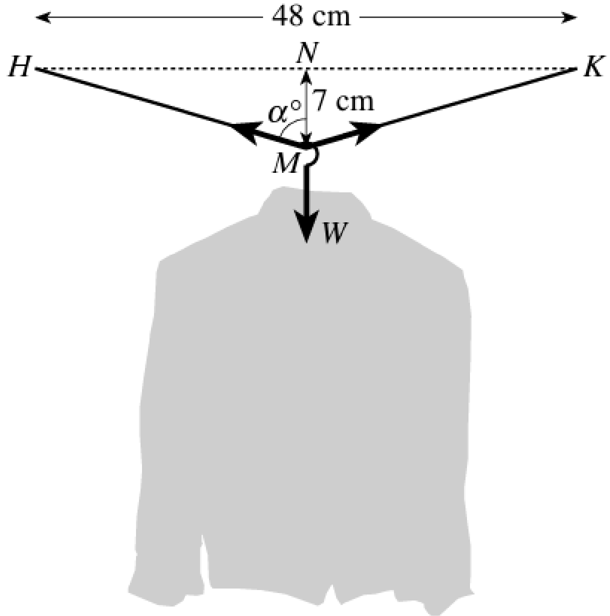


$$l = 0.4 \text{ m} \quad \text{(Natural length)}$$

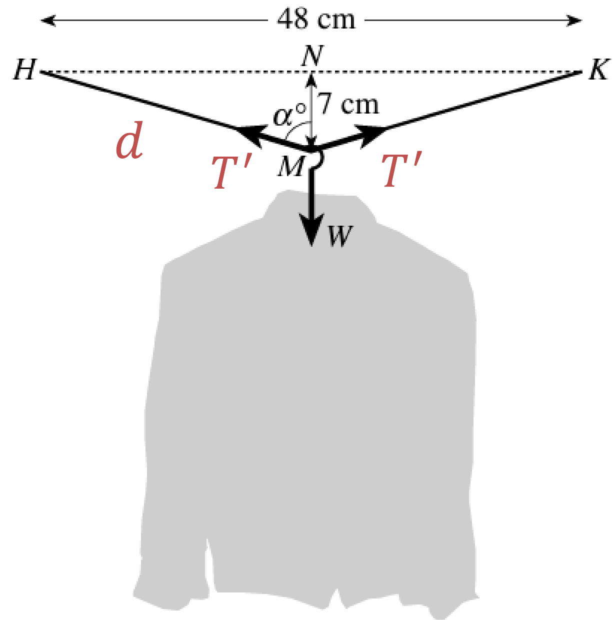
$$x = 0.48 - 0.4 = 0.08 \text{ m}$$

$$\therefore T = \frac{\lambda x}{l} \Rightarrow 10 = \frac{\lambda(0.08)}{0.4}$$

$$\therefore \lambda = 50 \text{ N}$$

Example 5.1.3

A student uses a 40 cm length of curtain wire as a washing line. One end is attached to a hook H on the wall. When he has some laundry to dry, he stretches the wire so that the other end reaches another hook K at the same level 48 cm away. The tension in the wire is then 10 newtons. When the student hangs a wet shirt at the mid-point M of the wire, the wire stretches further so that M is 7 cm below the mid-point N of HK. How much does the wet shirt weigh?



$l = 0.4$ m $\qquad \lambda = 50$ N

$$\alpha = \operatorname{atan}\left(\frac{24}{7}\right) = 73.74°$$

$$d = \sqrt{24^2 + 7^2} = 25 \text{ cm} = 0.25 \text{ m}$$

$$x' = 2(0.25) - 0.4 = 0.1 \text{ m (extension)}$$

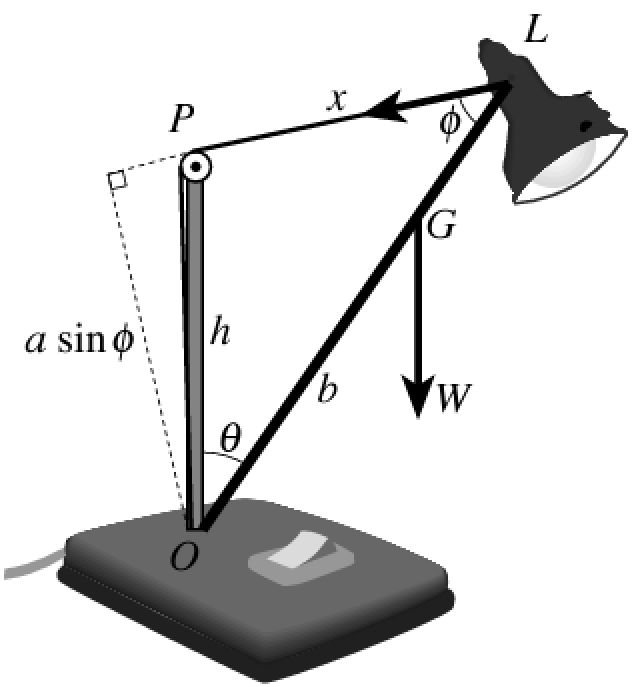
$$T' = \frac{\lambda x'}{l} = \frac{50(0.1)}{0.4} = 12.5 \text{ N}$$

$$+\uparrow)\ F_y = 0: \quad 2T' \cos\alpha - W = 0$$

$$\therefore W = 2(12.5)\cos 73.74° = 7.0 \text{ N}$$

Example 5.1.4

Fig. 5.3 shows the design for an adjustable desk lamp in which the light bulb can rest in various positions. O is a point of the base, and a vertical column fixed at O has a small pulley P at the top. A rigid arm OL is hinged at O so that it can rotate in a vertical plane. A lamp is attached to the arm at L. An elastic cord passes over the pulley, with one end fixed at O and the other end fixed to the arm at L. The natural length of the cord is equal to the height of the column OP, so that the extension of the cord is equal to PL.

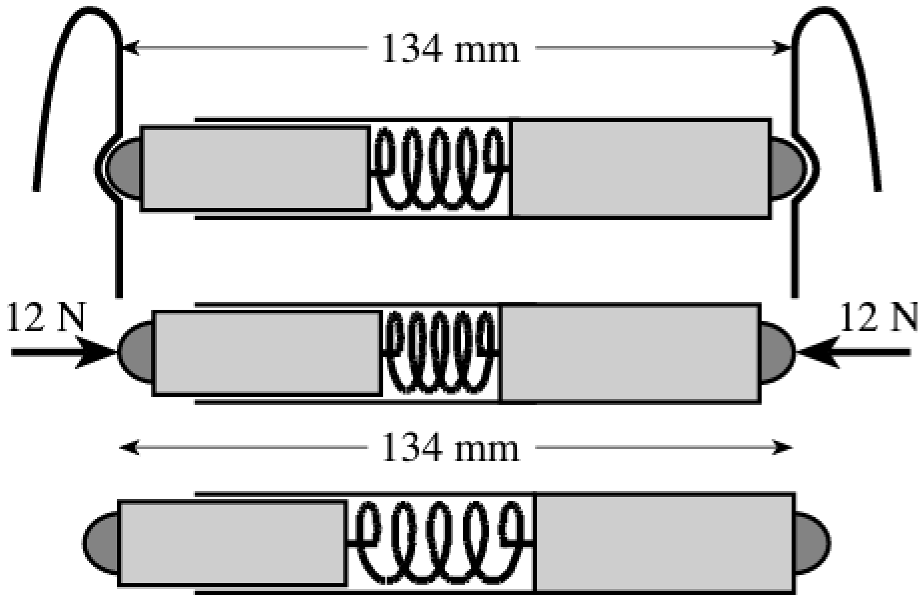


Self study!

5.2 Springs and rods

Example 5.2.1

Fig. 5.5 shows the design of the bar in a toilet-roll holder, which fits between the two arms of a metal frame 134 mm wide. When out of the frame, the cylindrical part of the bar just fits between the two arms. The projections at the ends are 6 mm long, and they fit into cavities 2 mm deep in the frame. To compress the bar so as to get it out of the frame requires a force of 12 newtons. How much force does the bar exert on each arm when it is in place?



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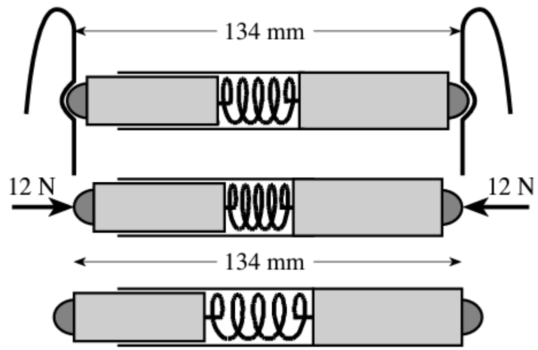


We do not know the length of the spring and also not the value of k

To compress 12 N: $x = 2 \times 0.006 = 0.012$ m

$\therefore T = kx \Rightarrow 12 = k(0.012)$

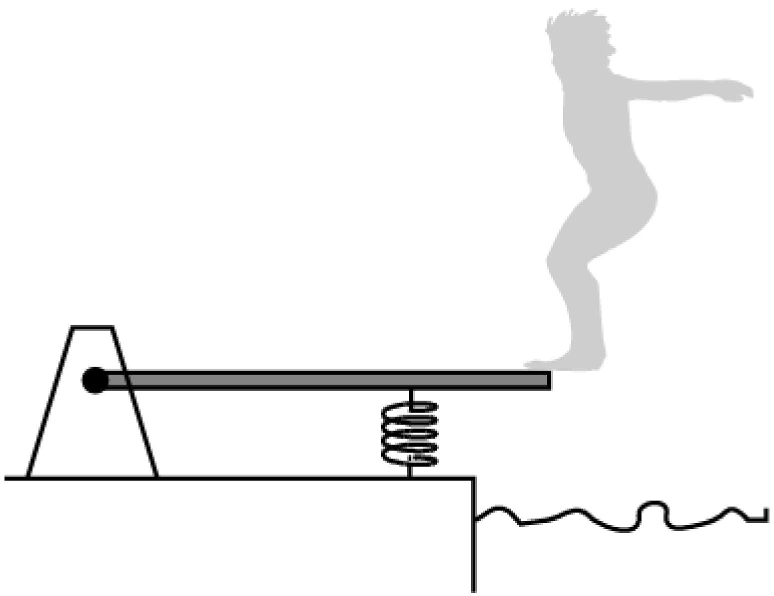
$\therefore k = 1000$ N/m

When in place: $x' = 2(0.006 - 0.002) = 0.008$ m

$\therefore T' = kx' = 1000(0.008) = 8$ N

Example 5.2.2

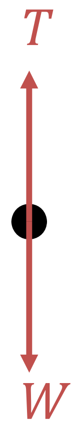
Fig. 5.6 shows a design for a diving board. The board is uniform, 2 metres long and has mass 35 kg. It is hinged at one end, and is supported in a horizontal position 50 cm above the edge of the pool by a spring 1.4 metres from the hinge. It is specified that, when a boy of mass 49 kg stands at the pool end of the board, that end should not go down by more than 2 cm. Modelling the board as rigid, calculate the natural length and the modulus of elasticity of the spring needed to just satisfy that condition.



Self study!

Class exercises: Exercise 5A pg. 84

2 A ring of mass 20 grams is lifted gently off a table at a steady speed by an elastic thread of natural length 28 cm and modulus of elasticity 7 newtons. What is the length of the thread while it is lifting the ring?



$$m = \frac{20}{1000} = 0.02 \text{ kg}$$

Steady speed $\Rightarrow a = 0$

$$+\uparrow)\ F_y = 0: \quad T - W = 0$$

$$\therefore \frac{\lambda x}{l} = W$$

$$\therefore x = mg\left(\frac{l}{\lambda}\right) = 0.02(10)\left(\frac{0.28}{7}\right) = 0.008 \text{ m}$$

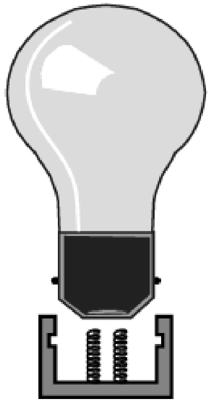
$$\therefore \text{length} = 0.28 + 0.008 = 0.288 \text{ m}$$

Class exercises
Exercise 5A pg. 84

4 A box of weight 20 newtons is placed on a table. It is to be pulled along by an elastic string with natural length 15 cm and modulus of elasticity 5 newtons. The coefficient of friction between the box and the table is 0.4. Holding the string horizontally by its loose end, and beginning with the string just taut, how far would you have to pull before the box starts to move?

Class exercises
Exercise 5A pg. 84

6 A new bulb with a bayonet fitting is to be inserted into a vertical lamp-holder. The mass of the bulb is 40 grams. To insert the bulb it is first placed in the holder so that it rests on the two vertical springs. It then has to be pushed down 6 mm against the springs and twisted; then, when it is released, it rises 2 mm and is held firm by the pins which fit into the slots in the holder. The maximum force you need to exert during the process is 2 newtons. Find the force holding the bulb in position.



Stage 1: Resting on the springs

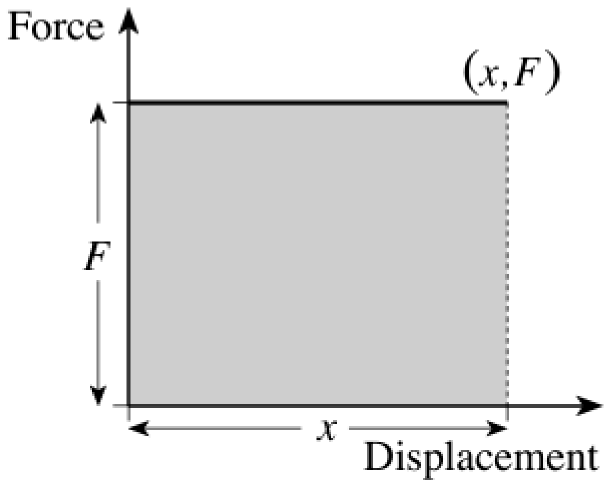
Stage 2: Pressed down (and twisted)

Stage 3: Lifts 2 mm and held in place

Self study	**Homework (to be handed in)**
Read pp. 86 – 89 Example 5.3.1 Example 5.4.1, 5.4.2 Exercise 5B:2 Miscellaneous Ex. 1, 7, 13, 17	Exercise 5B: 2 Miscellaneous Ex. 1, 7, 17

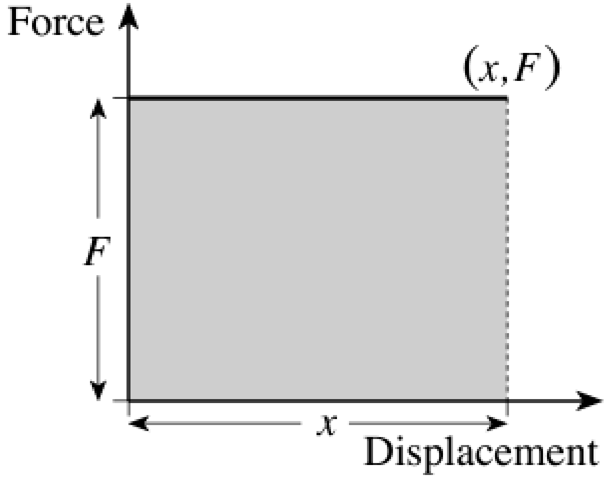
5.3 Work done

5.3 Work done

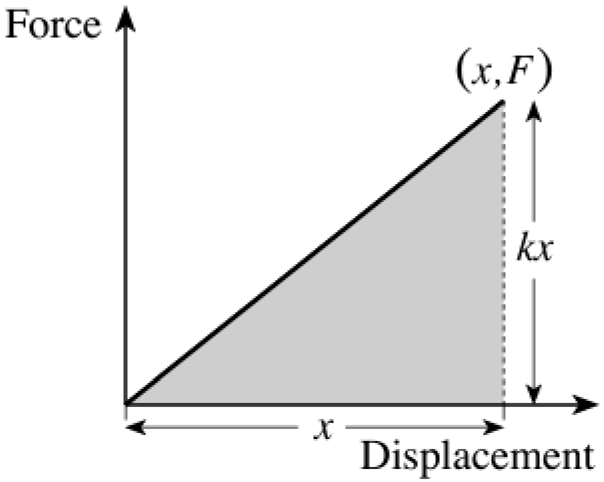


$$W = Fx$$

5.3 Work done in stretching and compressing



$$W = Fx$$



$$W = \frac{1}{2}kx^2$$

Final tension

$$W = \frac{1}{2} \times kx \times x$$

Extension

When a string is stretched, or a spring is stretched or compressed, starting at its natural length l, the work done in changing its length by an amount x is

$$\frac{1}{2}kx^2 = \frac{\lambda x^2}{2l},$$

where k is the stiffness and λ is the modulus of elasticity. This is $\frac{1}{2} \times \text{final tension} \times \text{extension}$, or $\frac{1}{2} \times \text{final thrust} \times \text{reduction in length}$.

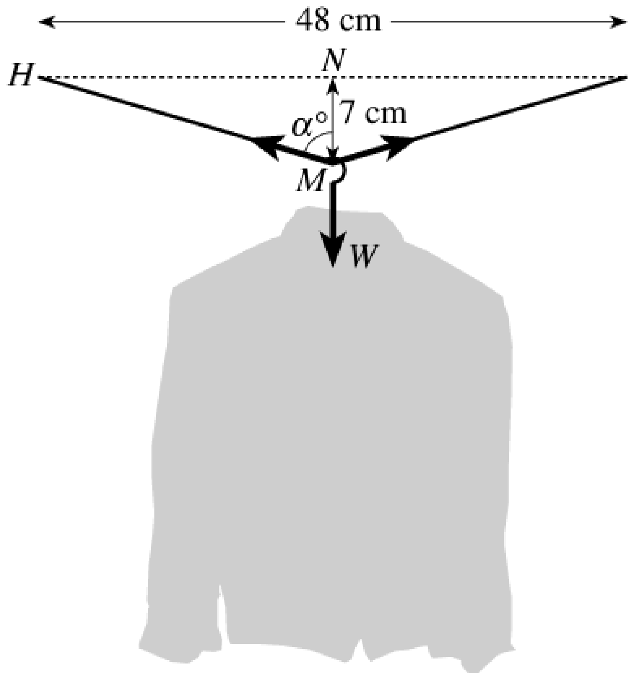
Example 5.3.1

In Example 5.1.3, how much work does the student do in fixing the curtain wire to the second hook?

Example 5.1.3

A student uses a 40 cm length of curtain wire as a washing line. One end is attached to a hook H on the wall. When he has some laundry to dry, he stretches the wire so that the other end reaches another hook K at the same level 48 cm away.

$l = 0.4$ m $\lambda = 50$ N



$$W = \frac{1}{2}kx^2 = \frac{\lambda x^2}{2l}$$

5.4 Elastic potential energy

A string or spring, stretched or compressed by a distance x, possesses **elastic potential energy** E of amount $E = \frac{1}{2}kx^2 = \frac{\lambda x^2}{2l}$.

If no work is done by non-conservative forces, the total energy (kinetic and potential, both gravitational and elastic) remains constant.

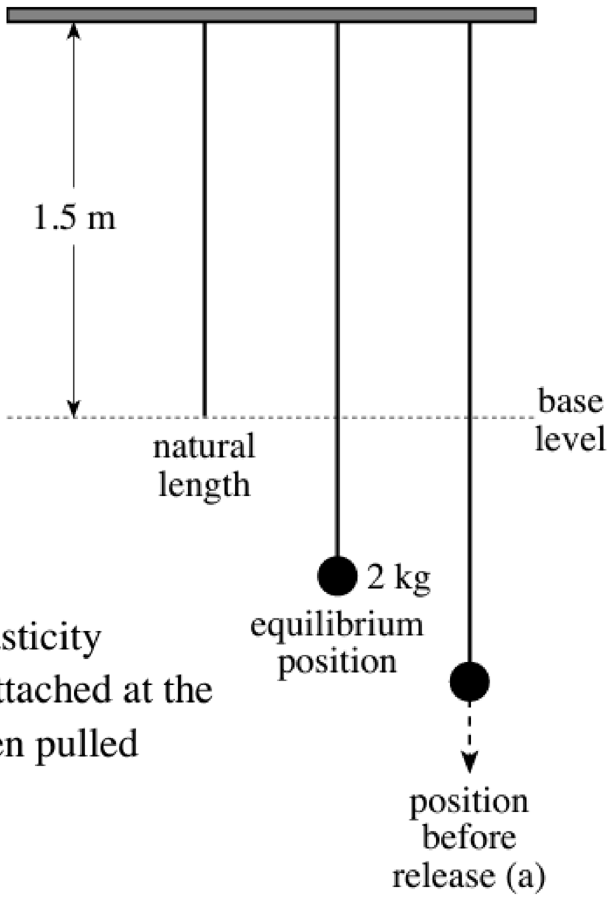
$$E_{k_1} + E_{P_1} = E_{k_2} + E_{P_1} \qquad \Delta E_k + \Delta E_p = 0$$

Kinetic energy: $E_k = \frac{1}{2}mv^2$

Potential energy: E_p

Example 5.4.1

In a toy gun a cork of mass 4 grams is shot out of the barrel by the release of a spring, which is compressed through a distance of 5 cm. A force of $6x$ newtons is needed to keep the spring compressed by x cm. Find the speed with which the cork leaves the barrel.



Example 5.4.2

One end of an elastic string, of natural length 1.5 metres and modulus of elasticity 50 newtons, is attached to a hook in the ceiling. A particle of mass 2 kg is attached at the other end, and hangs in equilibrium, as shown in Fig. 5.9. The particle is then pulled down a distance of b metres, and released. Find how high it rises

(a) if $b = 0.4$, (b) if $b = 0.9$.

Self study!

Class exercises
Exercise 5B pg. 90

3 An elastic string has natural length 2 metres and modulus of elasticity 45 newtons. One end is fastened to a fixed point O on a smooth table. A particle of mass 0.1 kg is attached to the other end. The particle is placed on the table 2.5 metres from O, with the string stretched, and then released. How fast is the particle moving when the string becomes slack?

How far is the particle from O when it next comes to rest? Describe the motion after that.

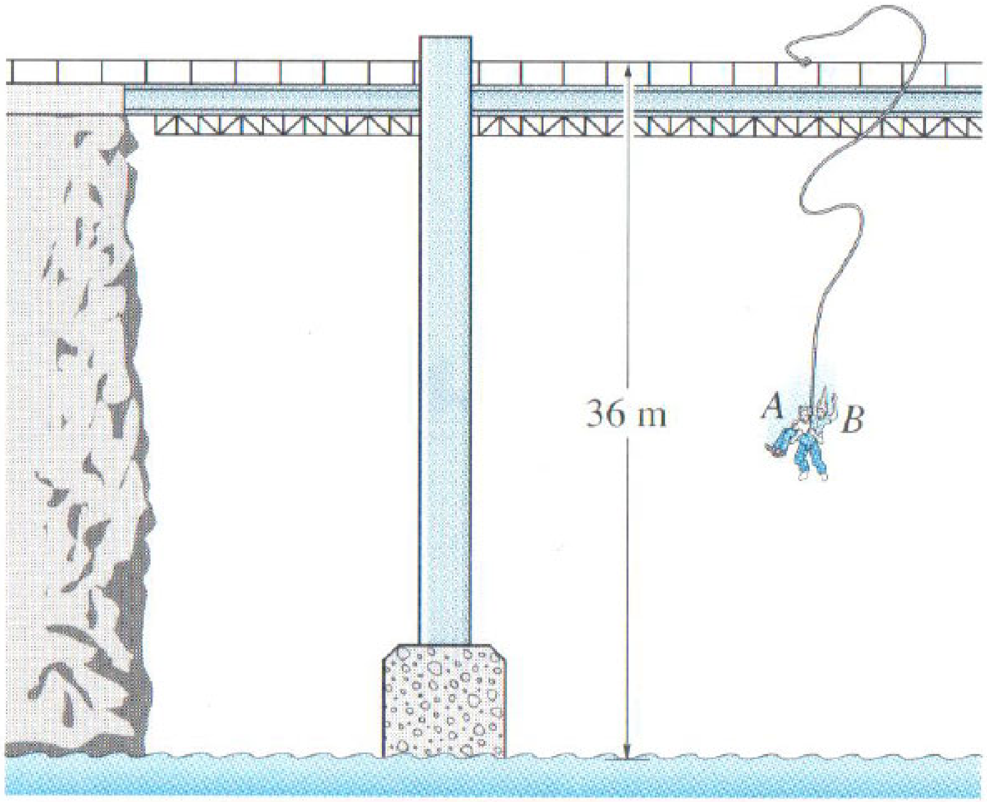
Miscellaneous Exercise 5 pg. 94

14 An elastic string is of unstretched length 20 cm and, when stretched by a distance x cm, has a tension of $\frac{1}{4}x$ newtons. Find an expression for the work done (in N cm) in stretching it by this distance.

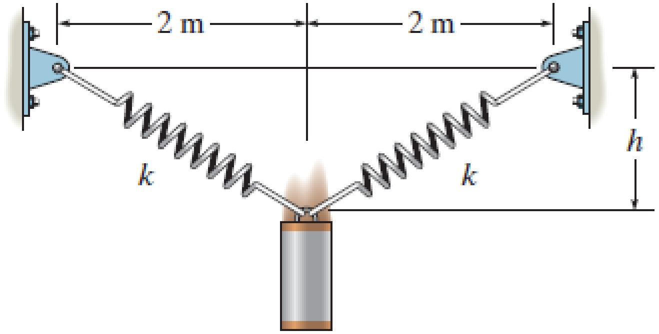
The string is fastened between two pegs A and B, 20 cm apart. Its mid-point, M, is pulled aside in a direction perpendicular to AB and is held a distance y cm from the mid-point of AB by a force of F newtons. Show that the string is stretched a distance $2\left(\sqrt{100+y^2}-10\right)$ cm. Hence show that $F = y - \dfrac{10y}{\sqrt{100+y^2}}$.

Find the work done by the force F newtons as y increases from 0 to 10. (OCR)

***14–88.** Just for fun, two 750-N engineering students A and B intend to jump off the bridge from rest using an elastic cord (bungee cord) having a stiffness $k = 1300$ N/m. They wish to just reach the surface of the river, when A, attached to the cord, lets go of B at the instant they touch the water. Determine the proper unstretched length of the cord to do the stunt, and calculate the maximum acceleration of student A and the maximum height he reaches above the water after the rebound. From your results, comment on the feasibility of doing this stunt.



If the 20-kg cylinder is released from rest at $h = 0$, determine the required stiffness k of each spring so that its motion is arrested or stops when $h = 0.5$ m. Each spring has an unstretched length of 1 m.



The spring has a stiffness $k = 200$ N/m and an unstretched length of 0.5 m. If it is attached to the 3-kg smooth collar and the collar is released from rest at A, determine the speed of the collar when it reaches B. Neglect the size of the collar.