Homework assignments

Self study	**Homework (to be handed in)**
Read pp. 144 – 149	Exercise 9A: 2, 9 Miscellaneous Ex. 9: Nr.2

9. Linear motion with variable forces

- Know how to describe the motion by a differential equation involving two or more of the quantities displacement, velocity and time
- Be able to solve the differential equation in cases where the variables are separable, and interpret the solution
- Know that acceleration can be expressed as $v\frac{\mathrm{d}v}{\mathrm{d}x}$
- Know that the work done by a force can be expressed as $\int F\,\mathrm{d}x$

9.1 Velocity-time equations

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

When can I use these equations?

ONLY OF THE ACCELERATION IS CONSTANT!

Example 9.1.1

A small car of mass $800\ \text{kg}$ is powered by an electric motor. The driver applies a gradually increasing accelerating force, given by $60\sqrt{t}$ newtons after t seconds. How long does the car take to reach its maximum speed of $10\ \text{m s}^{-1}$?

Example 9.1.2

A cyclist and her bicycle have total mass $100\ \text{kg}$. She is working at constant power of 80 watts. Calculate how long it takes her to accelerate from $4\ \text{m s}^{-1}$ to $8\ \text{m s}^{-1}$ along a level road,

(a) if air resistance is neglected,

(b) making allowance for air resistance of $0.8v$ newtons when her speed is $v\ \text{m s}^{-1}$.

9.2 Displacement-time equations

Example 9.1.1

A small car of mass 800 kg is powered by an electric motor. The driver applies a gradually increasing accelerating force, given by $60\sqrt{t}$ newtons after t seconds. How long does the car take to reach its maximum speed of 10 m s^{-1}?

$$v = \frac{1}{20}t^{3/2} \qquad t = 34.2 \text{ s}$$

Example 9.2.1

In Example 9.1.1, how far does the car travel before it reaches its maximum speed?

Example 9.1.2

A cyclist and her bicycle have total mass 100 kg . She is working at constant power of 80 watts. Calculate how long it takes her to accelerate from $4\ \text{m s}^{-1}$ to $8\ \text{m s}^{-1}$ along a level road,

(a) if air resistance is neglected,

$$v^2 = 1.6t + 16$$

(b) making allowance for air resistance of $0.8v$ newtons when her speed is $v\ \text{m s}^{-1}$.

$$t = 62.5[\ln 84 - \ln(100 - v^2)]$$

Example 9.2.2

In Example 9.1.2, how far does the cyclist travel in increasing her speed from $4\ \text{m s}^{-1}$ to $8\ \text{m s}^{-1}$

(a) if air resistance is neglected,

(b) making allowance for air resistance of $0.8v$ newtons when her speed is $v\ \text{m s}^{-1}$?

Class exercises
Exercise 9A pg. 133

1 A car of mass $1200\ \text{kg}$ is at rest on a level road. Two people push it, producing a total force given by $(240 - 12t)$ newtons, where t is the time in seconds, until this becomes zero after 20 seconds. How fast is the car then moving, and how far does it move while it is being pushed?

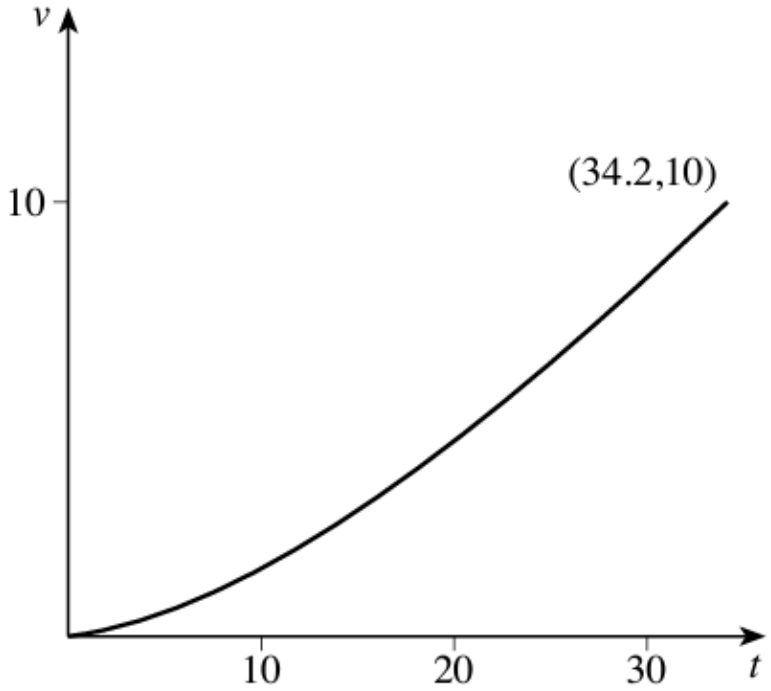
7 An aircraft of mass $4000\ \text{kg}$ lands on the deck of an aircraft carrier with a speed of $50\ \text{m s}^{-1}$. It is brought to rest with the help of air brakes and a parachute, which slow it down with a resisting force of $50v^2$ newtons, where v is the speed in m s^{-1}. Find how long it takes for the speed to drop to $10\ \text{m s}^{-1}$, and how far the aircraft travels in this time. Why shouldn't the aircraft rely on this means alone to bring it to rest?

9.3 Velocity-displacement equations

Example 9.1.1

A small car of mass 800 kg is powered by an electric motor. The driver applies a gradually increasing accelerating force, given by $60\sqrt{t}$ newtons after t seconds. How long does the car take to reach its maximum speed of 10 m s^{-1}?

$$v = \frac{1}{20}t^{3/2}$$

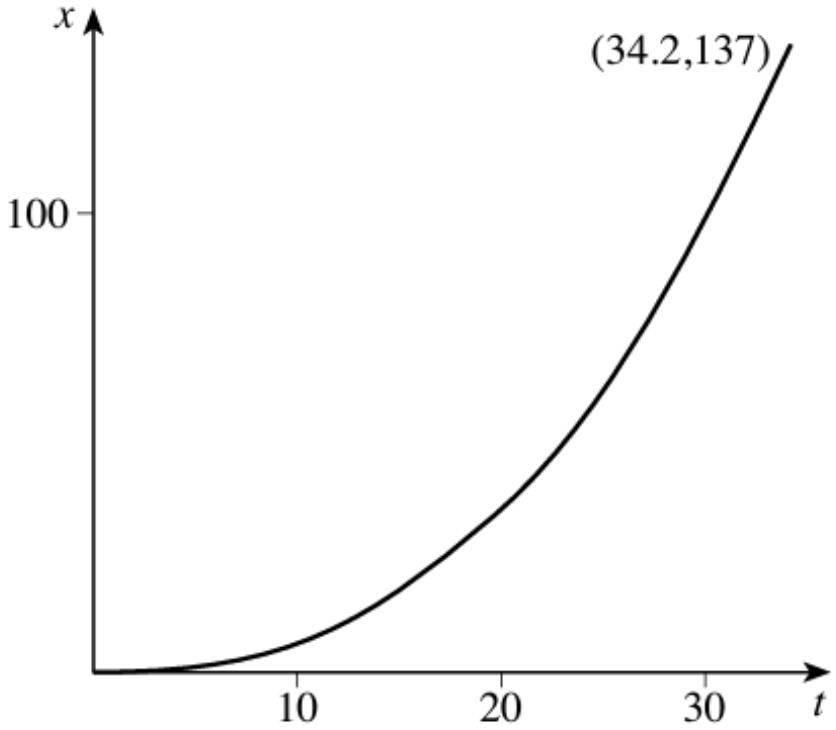


9.3 Velocity-displacement equations

Example 9.2.1

In Example 9.1.1, how far does the car travel before it reaches its maximum speed?

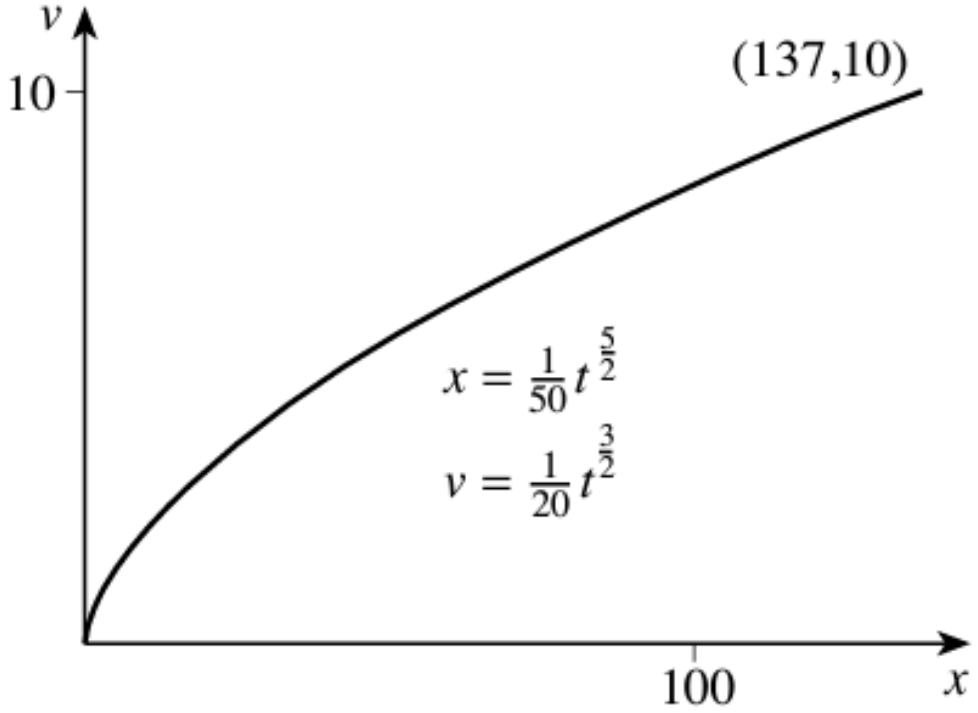
$$x = \frac{1}{50} t^{5/2}$$



$$v = \frac{1}{20}t^{3/2}$$

$$x = \frac{1}{50}t^{5/2}$$

$$v = 0.5228x^{3/5}$$



For an object moving in a straight line, if x denotes the displacement from a fixed point O of the line, v denotes the velocity and a the acceleration, then $a = v\dfrac{dv}{dx}$.

Example 9.3.1

A particle moves along a straight line in such a way that the velocity when it has travelled a distance x is given by $v = \dfrac{1}{p+qx}$, where p and q are constants. Find expressions for the acceleration (a) in terms of x, (b) in terms of v.

Example 9.3.2

A projectile is launched vertically upwards from the surface of the moon with initial speed u. The radius of the moon is R. When the projectile is at a height x above the surface, the gravitational attraction produces an acceleration $\dfrac{C}{(R+x)^2}$ towards the centre of the moon, where C is a positive constant. Find an expression for the speed of the projectile when it is at height x. How large must u be for the projectile never to return to the surface?

Self study

Example 9.1.2

A cyclist and her bicycle have total mass 100 kg . She is working at constant power of 80 watts. Calculate how long it takes her to accelerate from $4\ \text{m s}^{-1}$ to $8\ \text{m s}^{-1}$ along a level road,

(a) if air resistance is neglected,

(b) making allowance for air resistance of $0.8v$ newtons when her speed is $v\ \text{m s}^{-1}$.

$$\frac{80}{v} = 100\frac{\mathrm{d}v}{\mathrm{d}t}.$$

$$\frac{80}{v} - 0.8v = 100\frac{\mathrm{d}v}{\mathrm{d}t}$$

Example 9.3.3

In Example 9.1.2, how far does the cyclist travel in increasing her speed from $4\ \text{m s}^{-1}$ to $8\ \text{m s}^{-1}$ in the two cases (a) and (b)?

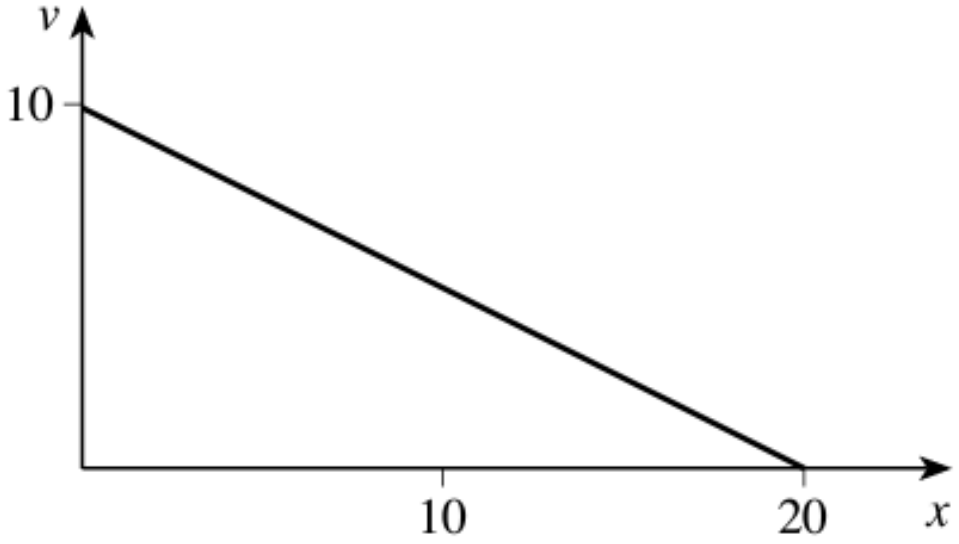
Self study

9.4 Reintroducing time

Example 9.4.1

A car is travelling at 10 m s^{-1} when the driver applies the brakes and brings the car to rest in a distance of 20 metres. The velocity–displacement relationship is modelled by a straight line graph. Find an expression for the distance the car has travelled t seconds after the brakes are applied.

$$x = 20\left(1 - e^{-\frac{1}{2}t}\right)$$



Check answer

9.5 Work done by a force

$$F = ma$$

$$\frac{1}{2}mv_1{}^2 + \int_{x_1}^{x_2} F\,dx = \frac{1}{2}mv_2{}^2$$

(Work) $W = \int_{x_1}^{x_2} F\,dx$

If an object modelled as a particle moving in a straight line changes its position from x_1 to x_2 under the action of a force F, the work done by the force is $\int_{x_1}^{x_2} F\,dx$.

Class exercises
Exercise 9B pg. 157

1 For the following equations and intervals, find equations connecting v with x. Draw sketches of the (t,x), (t,v) and (x,v) graphs.

(a) $x = 4t^2 + t$ for $0 \leqslant t \leqslant 2$

(b) $x = 3\sin 2t$ for $0 \leqslant t \leqslant 2\pi$

(c) $x = 5e^{-2t}$ for $0 \leqslant t \leqslant \ln 2$

(d) $x = \dfrac{1}{t+1}$ for $0 \leqslant t \leqslant 4$

2 A particle is at the origin at time $t = 0$, and its velocity is given by the following equations. Find equations connecting v with x, and find expressions for the acceleration

(i) in terms of x, (ii) in terms of v.

(a) $v = \sec^2 t$ for $0 \leqslant t \leqslant \frac{1}{2}\pi$

(b) $v = e^{\frac{1}{4}t}$ for $t \geqslant 0$

(c) $v = \sin \frac{1}{2}t$ for $t \geqslant 0$

(d) $v = e^t + e^{-t}$ for $t \geqslant 0$

7 A motorcycle with its rider has mass $300\ \text{kg}$. The power of the engine is $5\ \text{kW}$, and the air resistance is given by $0.5v^2$ newtons when the speed is $v\ \text{m s}^{-1}$. Find how far it travels in increasing its speed from $5\ \text{m s}^{-1}$ to $15\ \text{m s}^{-1}$.

Homework assignments

Self study	**Homework (to be handed in)**
Start working through the exam papers and pose questions via email	Exercise 9B: 1 (a) 2(b). 9 **Hand in on:**

9.6 Vertical motion with air resistance

Example 9.6.1

A cannonball is projected vertically upwards from a mortar with an initial speed of $40\ \mathrm{m\,s^{-1}}$. The mortar is situated at the edge of a cliff 100 metres above the sea. On the way down, the cannonball just misses the cliff. In vertical fall the cannonball would have a terminal speed of $50\ \mathrm{m\,s^{-1}}$. Assuming that air resistance is proportional to the square of the speed, find how high the cannonball rises and how long it is in the air before falling into the sea.

Self study

Example 9.6.2

Rework Example 9.6.1, assuming that air resistance is proportional to the speed.

Example 9.6.1

A cannonball is projected vertically upwards from a mortar with an initial speed of $40\ \mathrm{m\,s^{-1}}$. The mortar is situated at the edge of a cliff 100 metres above the sea. On the way down, the cannonball just misses the cliff. In vertical fall the cannonball would have a terminal speed of $50\ \mathrm{m\,s^{-1}}$. Assuming that air resistance is proportional to the square of the speed, find how high the cannonball rises and how long it is in the air before falling into the sea.

Self study

Class exercises
Miscellaneous exercise 9 pg. 164

1 A particle P of mass 0.4 kg is initially at rest at a point O on a horizontal surface. In a simple model it is assumed that the surface is smooth. A horizontal force of magnitude $2t$ newtons is applied to P, where t seconds is the time after the force first begins to act. Given that the force is always directed away from O and that P moves in a straight line, calculate the speed of P when $t = 3$.

A more realistic model takes account of friction. State the effect of this on the initial behaviour of P. (OCR)

3 A particle of mass 2 kg is acted on by a single force of magnitude $8x$ newtons, where x metres is its displacement from a fixed point O. The force is directed away from O. The particle is at rest when it is 2 metres from O.

(a) Show that $v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 4x$, where v m s^{-1} is the velocity of the particle when its displacement is x metres.

(b) Find v in terms of x. (OCR)

10 A car of mass $1200\ \text{kg}$ is travelling on a straight horizontal road, with its engine working at a constant rate of $25\ \text{kW}$. Given that the resistance to motion of the car is proportional to the square of its velocity and that the greatest constant speed the car can maintain is $50\ \text{m s}^{-1}$, show that $125\,000 - v^3 = 6000v^2 \dfrac{\mathrm{d}v}{\mathrm{d}x}$, where $v\ \text{m s}^{-1}$ is the velocity of the car when its displacement from a fixed point on the road is x metres. Hence find the distance covered by the car in increasing its speed from $30\ \text{m s}^{-1}$ to $45\ \text{m s}^{-1}$, giving your answer to the nearest metre. (OCR)

13 Particles of mass 1.5 kg and 0.5 kg are attached to the ends of a light inextensible string which passes over a fixed smooth peg. The system is released from rest in a position where both parts of the string hanging from the peg are vertical. Assume that there is a resisting force on each particle due to its motion through the air and that the magnitude of this force is $\frac{1}{40}v^2$ newtons, where v m s^{-1} is the speed of the particle when it has moved a distance x metres from its original position. Show that $40v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 200 - v^2$, and find v in terms of x.

Show that the tension T newtons in the string is given by $T = 10\left(1 - \frac{1}{4}\mathrm{e}^{-0.05x}\right)$.

(OCR, adapted)

6 A cyclist and her bicycle have a total mass of 60 kg. The cyclist rides in a horizontal straight line, and exerts a constant force in the direction of motion of 150 N. The motion is opposed by a resistance of magnitude $12v$ N, where v m s^{-1} is the cyclist's speed at time t s after passing through a fixed point A.

(i) Show that $5\frac{dv}{dt} = 12.5 - v$. [2]

(ii) Given that the cyclist passes through A with speed 11.5 m s^{-1}, solve this differential equation to show that $v = 12.5 - e^{-0.2t}$. [4]

(iii) Express the displacement of the cyclist from A in terms of t. [3]